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DISCRETE GENERATORS ON PSEUDOTREES

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Dedicated to the memory of Peter J. Nyikos.

Abstract. We study pseudotrees endowed with a Sorgenfrey type topology. For a point t, we define its valence as the minimum size of a maximal antichain in the set of successors of t, and we use this to distinguish discrete from indiscrete branching. In the discretely branching case we define immediate branching and show that every point will have a set of immediate successors. We then introduce pseudo-suprema valence, which requires that the minimal upper bounds of every bounded chain to be valent with respect to all upper bounds. For pseudotrees with the Sorgenfrey topology, Dedekind completeness implies Hausdorff, zero-dimensional, and Tychonoff separation properties, and under pseudo-suprema valence these properties are equivalent. For immediately branching pseudotrees, pseudo-suprema valence is equivalent to the nonexistence of an embedded copy of ω^* in the successors of any point. Finally, for Hausdorff, immediately branching, pseudo-suprema valent pseudotrees, we characterize the existence of a super generator in terms of mutually agreeing clopen local bases and cofinal sequences.

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1. Introduction

Pseudotrees are a natural relaxation of trees. Instead of every set of predecessors being well-ordered, pseudotrees require the set of predecessors to only be totally ordered. Pseudotrees have a history in logic and order theory regarding the study of the pseudo tree-algebra (the initial chain algebra) and its associated Stone space as seen in [2], [6], and [1]. However, much investigation into the structure and properties intrinsically related to topologies on pseudotrees is lacking. The goal of this paper is to first, understand what global or local requirements may be placed on pseudotrees in order for trees to be an emergent subclass, and second, classify all pseudotrees which admit a discrete $(0, 1)$-generator such that the image of each function is the two-point $\{0, 1\}$.

The second section is dedicated to classifying pseudotrees based on their branching. To do this, we define valent sets which are maximal antichains in the successors to a point such that every successor is comparable to one point in the valent set. A pseudotree will be discretely branching if such a valent set exists, and be indiscretely branching otherwise. After, we classify pseudotrees where the partitions induced by valent sets admit minimal elements as immediately branching.

The third section is devoted to studying the local structure of immediately branching pseudotrees and their topological properties when equipped with the Sorgenfrey topology (upper-limit topology). It will be shown that the classical equivalence of Dedekind completeness, zero-dimensionality, Hausdorff, and Tychonoff, which is witnessed in trees, can be recovered in immediately branching pseudotrees if the minimal pseudo-suprema of a every bounded chain form a valent set. Then, we give an equivalence of immediately branching pseudotrees as those which do not admit an embedding of the reverse ordering of ω. Lastly, we define the tools needed to study to generators on pseudotrees.

The final section gives a classification theorem for immediately branching pseudotrees to admit discrete $(0, 1)$-generators such that the image of any function is precisely $\{0, 1\}$. The theory of discrete $(0, 1)$-generators was recently discussed in [4]. We show that these generators determine a collection of cofinal sequences and a collection of local bases. We show that an immediately branching Hausdorff pseudotree will have a super generator if and only if there exists a unified pair of clopen local bases and cofinal sequences. Lastly, we offer a few immediate corollaries relating to the existence of super generators on trees, and fully extending the result in [4] showing that every ordinal has a discrete $(0, 1)$-generator.

2. Psuedotrees, Valence, and Branching

In this section, we review definitions pertaining to partial orderings. Then, we define valent sets as maximal antichains of such that every element is comparable to exactly one member of the antichain. Then, we show that pseudotrees that admit valent sets above points such that the comparability equivalence class for each point in the valent set has a minimal element are locally trees.

2.1. **Conventions and Pseudotrees.**

All topological and set-theoretic definitions are standard and can be found in [3], [7], and [5]. Only ZFC is assumed. $\alpha, \beta, \gamma, \ldots$ are ordinals, $\kappa, \lambda, \mu, \nu \ldots$ are cardinals, ω is the least infinite limit ordinal. If α is an ordinal, α^* denotes α equipped with the reverse order.

Let $(X, \leq)$ be a poset. Elements $x, y \in X$ are said to be *comparable* if either $x \leq y$ or $y \leq x$, and x and y are said to be *incomparable* otherwise. We will make use of the following notation throughout: for a fixed $x \in X$,

$$x^{<} := \{y \in X : y < x\} \qquad x^{\leq} := \{y \in X : y \leq x\}$$
$$x^{>} := \{y \in X : y > x\} \qquad x^{\geq} := \{y \in X : y \geq x\}$$

An element $x \in X$ is *minimal* if $x^{<} = \emptyset$ and *maximal* if $x^{>} = \emptyset$.

If $Y \subset X$ is any subset, then Y is itself a poset with the inherited order from X. If $Y \subset X$, an *upper bound* (resp. *lower bound*) for Y is an element $x \in X$ such that $y \leq x$ (resp. $x \leq y$) for all $y \in Y$. A subset $C \subset X$ is called a *chain* if C is totally ordered. A subset $A \subset X$ is called an *antichain* if the elements of A are pairwise incomparable. X is *Dedekind complete* if every non-empty chain that is bounded above has a least upper bound.

A *tree* is a partially ordered set $(T, \leq)$ such that for all $t \in T$, $t^{<}$ is well-ordered. All trees will be assumed to have a minimal point called the *root*. For every $t \in T$, $h(t)$ is the order type of the set $t^{<}$ and is called the *height of* t. The *height of* T is defined to be

$$h(T) := \sup\{h(t) + 1 : t \in T\}.$$

For trees, $t^{+} = \{s \in T : t \leq s, h(s) = h(t) + 1\}$ is the set of immediate successors, and, if $h(t)$ is a successor ordinal, then t^{-} is the immediate predecessor of t.

Definition 2.1. A *pseudotree* is a partially ordered set $(T, \leq)$ such that $t^{<}$ is a chain for every $t \in T$.

Example 2.2. All ordinals, trees, and the real numbers with their usual ordering are pseudotrees.

2.2. **Valence of a Pseudotree.** We formalize branching at points in pseudotrees. We classify pseudotrees based on the two types of branching that they can exhibit. We call the two types *discrete branching* and *indiscrete branching*. We prove that if pseudotrees branch similarly to trees, then the set of successors at each point can be partitioned by their comparability to an invariant set of points which represent the direction of branching.

Definition 2.3. A point t in a pseudotree T is a *branch point* if there exist distinct points $t_1, t_2 \in t^>$ such that t_1 and t_2 are incomparable.

Definition 2.4. Let t be a point in a pseudotree T. The valence at t is defined to be

$$\lambda_t = \min\{|F| : F \text{ is a maximal antichain in } t^>\}.$$

Remark 2.5. If t is a branch point in T, Zorn's lemma guarantees the existence of at least one maximal antichain in $t^>$, hence $\lambda_t \geq 1$.

Proposition 2.6. *Let t be a point in a pseudotree T. t is maximal if and only if $\lambda_t = 0$. If t is not maximal and not a branch point, then $\lambda_t = 1$.*

Proof. If t is maximal, then $t^> = \emptyset$. Hence, the only maximal antichain is $C = \emptyset$, and $\lambda_t = 0$. If $\lambda_t = 0$, then the empty set is a maximal antichain in $t^>$. Hence $t^> = \emptyset$, and t is maximal.

If t is not maximal and not a branch point, then $t^>$ contains no two incomparable elements. Hence $t^>$ is a chain. Thus, maximal antichains are singletons, and $\lambda_t = 1$. □

Definition 2.7. Let X be a poset and let $F \subset X$ be a maximal antichain. We say that F is *valent with respect to X* if each $x \in X$ is comparable to a unique $f \in F$.

Definition 2.8. Let t be a point in a pseudotree T. We say $F \subset T$ is *valent at t* if it is valent with respect to $t^>$.

Definition 2.9. A point t is said to be *discretely branching* if there exists a maximal antichain F which is valent at t, and is said to be *indiscretely branching* otherwise.

Now, we prove the fact that valent sets for points which are discretely branching capture the notion of branching found in pseudotrees.

Proposition 2.10. *Let t be a discretely branching point in a pseudotree T. If F_1 and F_2 are two sets which are valent at t, then $|F_1| = |F_2|$.*

Proof. We prove this by constructing two injections $f : F_1 \to F_2$ and $g : F_2 \to F_1$ and invoking Cantor-Schröder-Bernstein. For $s \in F_1$, let

$f(s) = s'$ where s' is the unique element of F_2 that s is comparable to. Similarly, define $g : F_2 \to F_1$ in the same fashion. Then, to see that f is injective, take two $s_1, s_2 \in F_1$, and assume $f(s_1) = f(s_2) = s'$. This means that both s_1 and s_2 are both comparable to s'. But F_1 is valent, which means there is a unique $s \in F_1$ such that s' is comparable to s. But this means that $s = s_1 = s_2$ as desired. By the same argument, g is injective, and we get $|F_1| = |F_2|$. □

Lemma 2.11. *Let t be a discretely branching point in a pseudotree T. If F is valent at t, then $|F| = \lambda_t$.*

Proof. If $\lambda_t = 0$, then t is maximal, and the only set which is valent at t is the empty set, and we are done.

Now assume that $t^>$ is non-empty. If F is a maximal antichain in $t^>$, then $\lambda_t \leq |F|$ by definition. Let G be any maximal antichain in $t^>$. Consider the map $f : G \to F$ where f is defined by $f(s) = s'$ where s' is the unique member of F such that s is comparable to s'.

Now we show f is surjective. Assume it were not, then there exists $s' \in F$ such that no element of G is comparable to s'. But G is a maximal antichain, so $G \cup \{s'\}$ would also be an antichain - a contradiction. Hence f is surjective, and $|F| \leq |G|$. Thus, $\lambda_t \leq |F|$ and for every maximal antichain G, $|F| \leq |G|$, hence $|F| \leq \lambda_t$, therefore $\lambda_t = |F|$ as desired. □

Definition 2.12. A pseudotree T is *discretely branching* if each non-maximal branch point is discretely branching.

Remark 2.13. If T is a tree, then T is discretely branching, and for every $t \in T$, t^+ is valent at t and $\lambda_t = |t^+|$.

Thus, pseudotrees are split in two classes: discretely branching and indiscretely branching. We have exhibited that discretely branching pseudotrees witness branching behavior exactly like trees while indiscretely branching is a novel property.

2.3. Immediately Branching Property.

We now characterize discretely branching pseudotrees in terms of whether or not the successors at a point are immediate. We show that if a pseudotree is discretely branching and witnesses immediate successors, then immediate successors must be valent. Hence, branching in these pseudotrees is almost identical to trees. All pseudotrees through the rest of this section are assumed to be discretely branching.

Definition 2.14. Let t be a non-maximal point in a pseudotree T. Let $\mathcal{F}_t = \{F \subset t^> : F \text{ is valent at } t\}$. For a fixed $F \in \mathcal{F}_t$, and every $s \in F$, define the set called the *branch towards s* as

$$B_t(s) = \{t' \in t^> : t' \text{ is comparable to } s\}$$

Proposition 2.15. *For a fixed* $F \in \mathcal{F}_t$*, the family* $\mathcal{B} = \{B_t(s) : s \in F\}$ *is a partition of* $t^>$*.*

Proof. For any $t' \in t^>$, there exists a unique $s \in F$ such that t' is comparable to s as F is valent at t. Thus, $\mathcal{B}$ covers $t^>$. Moreover, let $s_1, s_2 \in F$. Assume $B_t(s_1) \cap B_t(s_2) \neq \emptyset$. Then, for any $t' \in B_t(s_1) \cap B_t(s_2)$, t' is comparable to both s_1 and s_2. But as F is valent, $s_1 = s_2$ and $B_t(s_1) = B_t(s_2)$. Thus, for any distinct $s_1, s_2 \in F$, $B_t(s_1) \cap B_t(s_2) = \emptyset$, and $\mathcal{B}$ is a disjoint family. □

Proposition 2.16. *For distinct* $F_1, F_2 \in \mathcal{F}_t$*, the families* $\mathcal{B}_1 = \{B_t(s) : s \in F_1\}$ *and* $\mathcal{B}_2 = \{B_t(s) : s \in F_2\}$ *are equal.*

Proof. Take $s_1 \in F_1$, and let $s_2 \in F_2$ be the unique point of F_2 that is comparable to s_1 guaranteed by valence. Take $r \in B_t(s_1)$, so r is comparable to s_1.

If $r \leq s_1$, then r is comparable to s_2. If $s_2 \leq s_1$, $s_1^{\leq}$ is a chain, and hence either $s_2 \leq r$ or $r \leq s_2$. If $s_1 \leq s_2$, then transitivity guarantees that $r \leq s_2$.

If $r \geq s_1$ and assume r is not comparable with s_2. Then, there exists $r' \in F_2$ such that r is comparable to r'. Since $s_1 \leq r$, then we have that s_1 is comparable to r' by the same argument above. However, s_1 is already comparable to s_2, so $s_2 = r'$, which is a contradiction.

In either case, we conclude that r is comparable to s_2 and $r \in B_t(s_2)$, and $B_t(s_1) \subset B_t(s_2)$. The reverse inclusion is the same as above, and we conclude that $B_t(s_1) = B_t(s_2)$.

As we have established that valent sets are in bijection with one another, we establish that $\mathcal{B}_1 = \mathcal{B}_2$. □

Definition 2.17. Let t be a non-maximal point in a pseudotree T, and let F be valent at t. Then, we say T *immediately branches at* t if for every $s \in F$, the branch towards s has a minimal element.

Remark 2.18. T being immediately branching at a point t is independent of choice of valent set F. This follows immediately from the previous proposition which shows that for any valent set, the partitions into the branches towards $s \in F$ are equal.

Proposition 2.19. *Let* t *be a point that immediately branches and has valence* λ_t *witnessed by a valent set* F*. Then, there is a set of immediate successors* t^+ *and* $\lambda_t = |t^+| = |F|$*.*

Proof. For each $s \in F$, let m_s be the minimal element of $B_t(s)$ guaranteed by immediately branching. Then, define $S = \{m_s : s \in F\}$. It is immediate that S is also valent at t. We claim that m_s is an immediate successor of t. If there were another $t < t' \leq m_s$, then t' is comparable to

s, and m_s is the least element of $B_t(s)$, so $m_s \leq t'$ and $t' = m_s$. Thus, m_s is an immediate successor, and $S = t^+$. Lastly, the previous proposition shows, as S is also valent, that $|F| = |S| = |t^+|$. $\square$

Definition 2.20. We say a discretely branching pseudotree T is *immediately branching* if and only if every non-maximal point $t \in T$ is immediately branching.

Corollary 2.21. *Let T be an immediately branching pseudotree. Then for every $t \in T$, the subset $\{t\} \cup t^+$ is a tree with height less than or equal to* 2*.*

We have thus characterized pseudotrees in terms of their branching properties. Immediately branching pseudotrees are locally pseudotrees at the level of immediate successors. Yet, immediately branching pseudotrees are still a much larger class of posets than trees themselves.

3. TOPOLOGICAL PROPERTIES OF PSEUDOTREES

In this section, we determine that to endow pseudotrees with the Sorgenfrey (upper-limit order) topology. To do so, we investigate how immediately branching pseudotrees differ from trees topologically. We will show that with an assumption about the existence of minimal upper bounds to chains, pseudotrees and trees with both necessitate that Hausdorff, Dedekind completeness, zero-dimensionality, and Tychonoff properties to be equivalent. Lastly, we prove that every immediately branching pseudotree with minimal upper bounds to chains will locally be a tree.

3.1. **The Sorgenfrey Topology.**

Definition 3.1. Let X be a poset. Define the collection

$$\mathcal{S} = \{x^> : x \in X\} \cup \{x^\leq : x \in X\}.$$

Then the topology generated by $\mathcal{S}$ as a subbase is called the *Sorgenfrey topology* and denoted $\tau_{\mathcal{S}}$.

Proposition 3.2. *Let T be a pseudotree endowed with the Sorgenfrey topology. For a fixed non-minimal $t \in T$ and any $s < t$, define the set $(s, t] = s^> \cap t^\leq$. Then $N(t) = \{(s, t] : s < t\}$ is a neighborhood base for t. If t is minimal, then t is isolated.*

Proof. Fix two finite collections of points $\{s_i\}_{i=1}^n$ and $\{t_j\}_{j=1}^m$, and consider the basic open set

$$B = \left(\bigcap_{i=1}^{n} s_i^>\right) \cap \left(\bigcap_{j=1}^{m} t_j^\leq\right)$$

formed from the subbase. Assume B is neighborhood of a non-minimal point $t \in T$.

As $t \in s_i^{>}$, we have that $s_i \in t^{\leq}$ for each i. As $t^{\leq}$ is a chain, and there are finitely many s_i, we can find a maximum. Call it $s = \max\{s_i : 1 \leq i \leq n\}$. Then, as $t \leq t_j$, $t^{\leq} \subset t_j^{\leq}$. Then clearly $(s, t] \subset B$.

If t is a minimal point in T, then take the basic open set $t^{\leq}$ from the subbase. By definition, $t^{\leq} = \{t\}$, and t is isolated. □

Proposition 3.3. *Let T be a pseudotree endowed with the Sorgenfrey topology. T is a T_0 space.*

Proof. Fix two distinct points $s, t \in T$. Either $s^{\leq}$ or $t^{\leq}$ will suffice to separate the points. If s and t are incomparable, either works. If $s \leq t$, $s^{\leq}$ works. If $t \leq s$, $t^{\leq}$ suffices. □

For the rest of the paper, every pseudotree will be assumed to be endowed the Sorgenfrey topology. An interesting fact about trees with this topology is the following

Theorem 3.4. *Let T be a tree. Then, the following are equivalent:*

i. *Hausdorff,*
ii. *Dedekind complete,*
iii. *zero-dimensional,*
iv. *Tychonoff.*

Thus, we determine a minimal condition on pseudotrees such the above properties also become equivalent.

Theorem 3.5. *Let T be a pseudotree endowed with the Sorgenfrey topology. If T is Dedekind complete, then T is also*

i. *Hausdorff,*
ii. *zero-dimensional,*
iii. *Tychonoff.*

Proof. Let T be a Dedekind complete pseudotree. We proceed in order.

Let $t_1, t_2 \in T$ be distinct points. If t_1 and t_2 are comparable, then, without loss of generality, let $t_1 < t_2$, then $U = t_1^{\leq}$ and $V = (t_1, t_2]$ are separating open neighborhoods.

Assume t_1 and t_2 are incomparable. If $t_1^{\leq}$ and $t_2^{\leq}$ are disjoint, then they are separating. Assume otherwise, and let $C = t_1^{\leq} \cap t_2^{\leq}$. Note that C is a bounded chain and must have a least upper bound $c \in T$. Then, the neighborhoods $U = (c, t_1]$ and $V = (c, t_2]$ are disjoint. To see this, assume $U \cap V$ is not empty, then there would be a r such that $r \in t_1^{\leq} \cap t_2^{\leq}$ and $r > c$, a contradiction of the upper bound of c.

We show that every $t \in T$ has a basis of clopen sets. If t is minimal, this is immediate as Hausdorff guarantees that singletons are closed. Assume t is not minimal, and let $s < t$ and $(s, t]$ a neighborhood of t. We will show that $s^{>}$ and $t^{\leq}$ are themselves closed.

Let $r \in (t^{\leq})^c$. If $r > t$, then $(t, r]$ is an open neighborhood of r missing $t^{\leq}$. If r and t are incomparable, then consider the set $C = t^{\leq} \cap r^{\leq}$. If C is empty, then $r^{\leq} \subset (t^{\leq})^c$. If C is not empty, then C is a bounded chain with a least upper bound c. Then $(c, r]$ is an open neighborhood of r missing $t^{\leq}$. Indeed, if $t^{\leq} \cap (c, r]$ were not empty, then there is some $c < c' \leq r$ and $c' \leq t$, but then c' would be in C, contradicting the least upper bound c. Therefore, $(t^{\leq})^c$ is open, and $t^{\leq}$ is clopen.

Let $r \in (s^{>})^c$. If $r \leq s$, then $s^{\leq}$ is a neighborhood of r missing $s^{>}$. If r and s are incomparable, then consider the set $C = r^{\leq} \cap s^{\leq}$. Again, if C is empty, then $r^{\leq} \subset (s^{>})^c$. If C is not empty, then it has a least upper bound c. Then $(c, r]$ will again be an open neighborhood of r disjoint from $s^{>}$.

Hence $s^{>}$ and $t^{\leq}$ are clopen, and $(s, t] = s^{>} \cap t^{\leq}$ is clopen, and T is zero-dimensional.

Every zero-dimensional Hausdorff space is completely regular. □

The issue here is very direct. In general, pseudotrees will not have the property that Hausdorff implies Dedekind complete.

Example 3.6. Let $T = \omega + \omega^* + (\omega + 1)$. T is not Dedekind complete as the chain $\omega \subset T$ does not have a least upper bound and is not Dedekind complete. Yet it is Hausdorff, zero-dimensional, Tychonoff, and immediately branching while also being not discrete.

The following condition will enable pseudotrees to make Hausdorff and Dedekind completeness equivalent.

Definition 3.7. Let T be a pseudotree. If C is a bounded chain, let $B(C) = \{t \in T : \forall c \in C, c \leq t\}$ be the set of upper bounds, and let $MB(C) = \{m \in B(C) : m \text{ is minimal in } B(C)\}$ be the set of all minimal upper bounds for C. Then, we say that T is *pseudo-suprema valent* if $MB(C)$ is valent with respect to $B(C)$.

Stated informally and proven later, pseudo-suprema valence necessitates no infinite descending upper bounds for any chain. It first needs to be noted that pseudo-suprema valence is not a rewording of Dedekind completeness.

Proposition 3.8. *Dedekind completeness is strictly stronger than pseudo-suprema valence.*

Proof. Let T be Dedekind complete. And let C be any bounded chain in T. Then, C is guaranteed to have a least upper bound $c \in T$. Then the set $F = \{c\}$ is valent with respect to $B(C)$ as each element $t \in B(C) \setminus \{c\}$ has the property that $c \leq t$.

However, there exists pseudo-suprema valent pseudotrees that are not Dedekind complete. The simplest example is a non-Dedekind complete tree like $\omega \cup \{\infty_1, \infty_2\}$ where $n < \infty_1$ and $n < \infty_2$ and ∞_1 and ∞_2 are incomparable. The set $\omega \subset T$ is a bounded chain, and the minimal upper bounds are $\{\infty_1, \infty_2\}$ which is trivially valent with respect to itself. Moreover, for any other chain in in $\omega \cup \{\infty_1, \infty_2\}$ if it bounded in ω, the minimal upper bound will be the maximum element. □

Lemma 3.9. *Let T be a Hausdorff and pseudo-suprema valent pseudotree. Then T is Dedekind complete.*

Proof. Let C be a bounded chain. Because T is pseudo-suprema valent, $MB(C)$ is not empty. By way of contradiction, assume that $t_1, t_2 \in MB(C)$ are distinct points. Since T is Hausdorff, choose disjoint basic neighborhoods $U = (s_1, t_1]$ and $V = (s_2, t_2]$ of t_1 and t_2 respectively. Because t_1 is a minimal upper bound of C, there exists $c_1, c_2 \in C$ such that $s_1 < c_1 \leq t_1$ and $s_2 < c_2 \leq t_2$.

As C is a chain, we may assume without loss of generality that $c_1 \leq c_2$. But, this implies that $c_2 \in U$, a contradiction. Therefore, $MB(C)$ is a singleton, say $\{c\}$. Since $MB(C)$ is valent in $B(C)$, every upper bound $u \in B(C)$ is comparable to c, and the minimality of c ensures that $c \leq u$. Thus, c is a least upper bound of C. □

Theorem 3.10. *Let T be a pseudo-suprema valent pseudotree. Then, the following are equivalent:*

i. *Hausdorff,*
ii. *Dedekind complete,*
iii. *zero-dimensional,*
iv. *Tychonoff.*

Proof. By the above lemma, every Hausdorff pseudo-suprema valent pseudotree is Dedekind complete by lemma 3.9. Every Dedekind complete pseudotree is Hausdorff, zero-dimensional, and Tychonoff by theorem 3.5. Every Tychonoff space is automatically Hausdorff. Lastly, every T_0 zero-dimensional space is Hausdorff. To see this, let $s, t \in T$ be distinct points. Since τ_S is T_0, take a basic clopen neighborhood U of t that separates s. Then U^c is a neighborhood of s and disjoint from U. □

3.2. **Recovering Trees from Pseudotrees.**

We finally characterize precisely when pseudotrees are locally trees. In doing so, we show exactly when immediately branching pseudotrees fail to be pseudo-suprema valent.

Theorem 3.11. *Let T be an immediately branching pseudotree. T is not pseudo-suprema valent if and only if there exists a point $t \in T$ such that ω^* embeds into $t^>$.*

Proof. Assume T is not pseudo-suprema valent. Then, there exists a bounded chain C such that $MB(C)$ is not valent with respect to $B(C)$. This reduces to two cases. Either $MB(C)$ is empty or there exists an upper bound that is not comparable to any minimal upper bound.

To see this, failure of the valency of $MB(C)$ with respect to $B(C)$ implies that there is some $u \in B(C)$ such that there does not exist a unique $s \in MB(C)$ such that u and s are comparable. Failure to find a unique element could happen in two ways: there are actually more than two $s_1, s_2 \in MB(C)$ such that u is comparable to both, or there are no elements in $MB(C)$ that u is comparable to. In the first case, if u is comparable to both, s_1 and s_2, and s_1 and s_2 are both minimal upper bounds, then this would require $s_1 \leq u$ and $s_2 \leq u$ and s_1 and s_2 incomparable. Hence $u^<$ would fail to be a chain. Thus, there is only one mode of failure of valency for non-empty $MB(C)$, and that is when there exists a $u \in B(C)$ that is incomparable to all of $MB(C)$.

First, assume $MB(C) = \emptyset$. Then, $B(C)$ has no minimal elements. Pick any $u_0 \in B(C)$. As u_0 is not minimal, $u_0^< \cap B(C)$ is not empty. Then, pick any $u_1 \in u_0^< \cap B(C)$. In general, if we have chosen u_n, then $(\bigcap_{i=1}^n u_i^<) \cap B(C)$ is non-empty, and pick any u_{n+1} from this set. Hence, we have an embedded ω^* into T. For any $c \in C$, as u_n are all upper bounds of C, then $c \leq u_n$. However, as $u_{n+1} < u_n$ strictly by choice, we must have $c < u_n$ for all n. Otherwise, if $c = u_n$ for some n, then then we would have $u_{n+1} < c$, a contradiction on the fact that $u_{n+1} \in B(C)$. Let $\phi : \omega^* \to B(C)$ by $\phi(n) = u_n$. Then, $\phi(\omega^*) \subset c^>$.

Assume now that $MB(C)$ is not empty. Let $u \in B(C)$ such that there are no elements of $MB(C)$ that are comparable to u. As u is not minimal, select any $u_0 \in u^< \cap B(C)$. Note, that u_0 also cannot be minimal. So, we may apply the same construction in the case when $MB(C) = \emptyset$ to obtain that there is some $c \in C$ such that there exists an order embedded ω^* in $c^>$.

Now, assume there is a point $t \in T$ such that ω^* embeds into $t^>$. Let $\{u_n : n < \omega\} \subset t^>$ be the image of ω^*. As T is immediately branching, the set t^+ exists and partitions $t^>$. So, there exists $s \in t^+$ such that $\{u_n : n < \omega\} \subset B_t(s)$.

Define the set $C = \{x \in B_t(s) : \forall n, x < u_n\}$. The set C is non-empty and a chain. It is not empty because $s \in C$, and is a chain because

$C \subset u_0^<$, and is bounded above as u_n is an upper bound for each $n < \omega$. So $B(C)$ is not empty. Then we claim that $MB(C)$ is not valent.

If $MB(C)$ was valent, then there would be a minimal upper bound m such that $m < u_n$ for each $n < \omega$. But as T is immediately branching, then m^+ exists. Moreover, there exists $m' \in m^+$ such $m' < u_n$ for every n because T is immediately branching and m^+ is valent. But, this would imply that m' is also in C, and $m < m'$ which contradicts the minimality. Therefore, $MB(C)$ cannot be valent, and T is not pseudo-suprema valent. □

The proof of the previous theorem actually highlights a stronger statement. In an immediately branching pseudotree, if you are able to embed ω^* into $t^>$ for some $t \in T$, the existence of immediate successors to t necessitates $\omega + \omega^*$ to be embeddable as well. However, we will not make use of such a fact.

Lastly, we make concrete the statement that immediately branching pseudo-suprema valent pseudotrees locally look like trees.

Theorem 3.12. *Let T be an immediately branching pseudo-suprema valent pseudotree. Then, for all $s, t \in T$ with $s < t$, the set $(s, t]$ is well ordered. Moreover, $t^\geq$ is a tree.*

Proof. Assume $(s, t]$ is not well-ordered, then we may freely pick an infinite descending sequence $\{t_n : n < \omega\}$ such that $t_{n+1} < t_n$. Hence, ω^* can be embedded into $s^>$ which is a contradiction.

Consider the set $t^\geq$ as subposet. We know that for any $t' \in t^\geq$, the set $(t, t']$ is well ordered. The set of predecessors of t' restricted to $t^\geq$ is precisely $[t, t')$, and is also well ordered as we are adding another least element and removing the maximum. Hence, for all $t' \in t^\geq$, the set of predecessors is well-ordered, and $t^\geq$ is a tree. □

Corollary 3.13. *Let T be a pseudotree. T is a tree if and only if it is an immediately branching pseudo-suprema valent pseudotree with a unique minimal element.*

Proof. The forward direction is immediate, and the reverse follows from the previous theorem as for the root r, $r^\geq = T$. □

3.3. **Limits, successors, and local bases.** Throughout the rest of the section, T is a Hausdorff, immediately branching, pseudo-suprema valent pseudotree. Before moving to the main result, we must prove some results detailing the local tree structure of these pseudotrees. As we are not guaranteed a minimal element of a pseudotree, we must adopt tree-like properties at every point.

Definition 3.14. A non-minimal point $t \in T$ has *relative limit height* if there exists an $s \in T$ with $s < t$ such that the height of t in $s^{\geq}$ is a non-zero limit ordinal. Denote the order type of t in the tree $s^{\geq}$ by $h_s(t)$.

Proposition 3.15. *If t has relative limit ordinal height with respect to some $s \in T$ with $s < t$, then it has relative limit ordinal height with respect to all $s' \in T$ such that $s' < t$.*

Proof. This follows from the ordinal arithmetic fact that for ordinals α, β, $\alpha + \beta$ is a limit ordinal if β is a non-zero limit ordinal.

If t has relative limit height with respect to $s < t$, then $h_s(t)$ is a limit ordinal. Consider any other s'. If $s' < s$, then $h_{s'}(t) = h_{s'}(s) + h_s(t)$ where the addition is taken as ordinal addition. Then $h_{s'}(t)$ is also a limit. Assume $s < s'$. Then, $h_s(t) = h_s(s') + h_{s'}(t)$. Then $h_{s'}(t)$ must also be limit. Otherwise if $h_{s'}(t) = \beta + 1$, then $h_s(t) = h_s(s') + (\alpha + 1) = (h_s(s') + \alpha) + 1$ would be a successor. ☐

From now on, we will not use "relative," and only say t is a *limit* or of *limit height*.

Proposition 3.16. *If t is a non-minimal point in T that does not have limit height, then t is an immediate successor.*

Proof. Since t is not minimal and does not have relative limit height, it does not have relative limit height for all $s < t$. Then, for any $s < t$, t has successor height in $s^{\geq}$. This means, there is some t' such that $t' < t$ and $t \in t'^{+}$. However, the fact that we have restricted ourselves to the poset $s^{\geq}$ does not change that t is an immediate successor of t' in T. ☐

The following propositions shows that immediate successors in pseudotrees behave precisely like immediate successors in pseudotrees. We can always find a limit point below an immediate successor where the immediate successor has finite height in the subtree.

Proposition 3.17. *If t is an immediate successor, and there is an $s < t$ such that s is of limit height, then there exists $s' < t$ such that s' is of limit height and $h_{s'}(t) < \omega$.*

Proof. Let t be an immediate successor and $s < t$ be a limit. Consider the chain $C = \{u < t : u \text{ limit}\}$. Then C is a non-empty and bounded by t^{-}. Hence, it has a least upper bound. If t_m is the least upper bound of C, then t_m is a limit. Assume otherwise that t_m is an immediate successor, then t_m^{-} would also be an upper bound for C. Hence t_m is a limit.

As t is an immediate successor, then consider the ordinal $h_{t_m}(t)$. If $h_{t_m}(t) > \omega$, then, there exists a $t' \in t_m^{\geq}$ such that $h_{t_m}(t) = \omega$. But then t would be in C. Thus, $h_{t_m}(t) < \omega$. ☐

Without a root in a pseudotree, hence making it a tree, we still must be able to define limits in terms of the smallest possible cofinal sequence that converges to them. Since t will not necessarily have a height in a pseudotree, we define a cardinal invariant which conceptualizes the accessibility to t.

Definition 3.18. Let t have limit height. Then, we define the *relative cofinality of* t, denoted $\mathrm{cf}_T(t)$, to be $\min\{\mathrm{cf}(h_s(t)) : s < t\}$. If t is a successor or minimal, use the convention that $\mathrm{cf}_T(t) = 1$.

Proposition 3.19. *Let t be a non-minimal point. If t is a limit, then there exists an increasing sequence $\{t_\mu : \mu < \mathrm{cf}_T(t)\}$ such that $t_\mu \to t$. If t is a successor or minimal, then t is isolated.*

Proof. Allow $s < t$ to be an element such that $\mathrm{cf}(h_s(t)) = \mathrm{cf}_T(t)$. Then consider the tree $s^{\geq}$. Let $\{\alpha_\mu : \mu < \mathrm{cf}_T(t)\} \subset h_s(t)$ be cofinal and $\alpha_\mu \to h_s(t)$. Then, let t_μ be the α_μth element of $t^{<}$.

To see that $t_\mu \to t$, take any basic open neighborhood $(t', t]$ of t. If $t' < s$, then $\{t_\mu : \mu < \mathrm{cf}_T(t)\} \subset (t', t]$. If $s < t'$, then we can find some t_ν such that $t' < t_\nu$ by cofinality. Hence, $\{t_\mu : \nu \leq \mu < \mathrm{cf}_T(t)\} \subset (t', t]$, we have $\{t_\mu : \mu < \mathrm{cf}_T(t)\}$ is eventually in $(t', t]$ as desired.

If t is an immediate successor, then let t^- be its predecessor. Then $(t^-, t] = \{t\}$. □

By the above, any tree can be partitioned into two sets, T', the derived set of T consisting of all points limits, and $T \setminus T'$, the set of all isolated points which are either minimal or immediate successors. Then, we are able to define local bases in terms of a limit points relative cofinality in a pseudotree.

Proposition 3.20. *Let t be a limit and $\{t_\mu : \mu < \mathrm{cf}_T(t)\}$ a monotonic increasing sequence with $t_\mu \to t$, then $\{(t_\mu, t] : \mu < \mathrm{cf}_T(t)\}$ is a local basis at t.*

Proof. For each $s < t$, and open neighborhood $(s, t]$, there exists a $t_\mu > s$ guaranteed by convergence, then $(t_\mu, t] \subset (s, t]$. □

Remark 3.21. Indeed, $\mathrm{cf}_T(t) = \chi(t) = \min\{|B_t| : B_t \text{ is a local basis at } t\}$, the character of t.

Now, we must intertwine two concepts: local bases and cofinal sequences at limit points. We start by defining collections of well behaved cofinal sequences.

Definition 3.22. Let t be a limit point. A mapping $\phi_t : \mathrm{cf}_T(t) \to t^{<}$ is said to be *limiting* if ϕ_t is order preserving and $\phi_t(\mu) \to t$ in T. If t is a successor, then $\phi_t : 1 \to t^<$ is said to be *limiting* if $\phi_t(0) = t^-$. And, if t is minimal, then $\phi_t : 1 \to \{t\}$ is said to be *limiting*.

Definition 3.23. A collection of maps $\mathcal{M} = \{\phi_t : t \in T\}$ is said to be *uniform limiting* if ϕ_t is limiting for each t.

Remark 3.24. Each uniform limiting collection of maps is equivalent to choosing a neighborhood basis of clopen sets for each $t \in T'$ defined in proposition 3.20.

As cofinal sequences determine local bases, it is important to note the concept when a limiting maps force points to be in their associated neighborhood.

Definition 3.25. Let t and t' be two points. We say a limiting function $\phi_{t'}$ *interacts* with t if there exists a $\mu < \mathrm{cf}_T(t')$ such that $\phi_{t'}(\mu) < t$. Let $\mathcal{M}$ be a uniform limiting collection. Then, we say $\mathcal{M}$ is *finitely interacting with* t if the set

$$\mathcal{M}(t) := \{s \in t^+ : \exists t' \in B_t(s), t' \text{ interacts with } t\}$$

is finite.

As will be revealed in next section, the local bases we will be mainly concerned with are those which consist of clopen sets and have property monotonic property of the usual half open interval base.

Definition 3.26. Any local basis of clopen sets $\mathcal{A}_t = \{A_{\alpha,t} : \alpha < \beta\}$ of a point t is *monotone* if $A_{\alpha+1,t} \subset A_{\alpha,t}$. A collection of clopen local bases $\mathcal{A} = \{\mathcal{A}_t : t \in T\}$ is *monotone* if each $\mathcal{A}_t$ is monotone.

Finally, we reveal the definition of what it means for a collection of limiting maps and a collection of clopen local bases to coherently interact with one another.

Definition 3.27. Let $\mathcal{M}$ be a uniform limiting collection and $\mathcal{A}$ be a monotone collection of local bases consisting of clopen sets. We call the pair $(\mathcal{M}, \mathcal{A})$ *homogeneous* if for each limit point t and $\mu < \mathrm{cf}_T(t)$, we have the relation

$$(\phi_t(\mu+1), t] \subset A_{\mu+1,t} \subset (\phi_t(\mu), t] \subset A_{\mu,t}.$$

Lastly, before turning our attention to generators, we need to define a property of clopen local bases. At this time, this property is not immediately clear why it needs to be defined. However, the idea is that we always want any collection of clopen local bases to have uniform behavior around the points they include.

Definition 3.28. Let $(\mathcal{M}, \mathcal{A})$ be a homogeneous pair. Then, $\mathcal{A}$ is said to be *agreeing above* t if for all $\mu < \mathrm{cf}_T(t)$, there exists a $F^{\mathrm{a}}_{\mu,t} \in [T]^{<\omega}$, such that for every $t' > t$ and $\nu < \mathrm{cf}_T(t')$, if $t \in A_{\nu,t'}$ and $\phi_{t'}(\nu) > t$, then $(A_{\mu,t} \Delta A_{\nu,t'}) \cap F^{\mathrm{a}}_{\mu,t} \neq \emptyset$.

Moreover, $\mathcal{A}$ is said to be *agreeing below* t if for every $\mu < \mathrm{cf}_T(t)$, there exists a $F^{\mathrm{b}}_{\mu,t} \in [T]^{<\omega}$, such that for every $t' > t$ and $\nu < \mathrm{cf}_T(t')$, if $\phi_{t'}(\nu) < t$, then $(A_{\mu,t} \Delta A_{\nu,t'}) \cap F^{\mathrm{b}}_{\mu,t} \neq \emptyset$.

Example 3.29. We will prove in section 4, that the collection of clopen local bases of the form $\mathcal{A}_t = \{(s,t] : s < t\}$ will always be agreeing above any point $t' < t$. So, we give an example of a collection that is not.

Let $T = \{n \leq 1 : n \in \mathbb{Z}\} \cup (\omega \times (\omega+1))$ where the ordering is given by the natural ordering for $n \in \mathbb{Z}$, and $\omega \times (\omega+1)$ will be branches emanating from 1. So, $\{n\} \times (\omega+1)$ will be ordered as $(n,i) \leq (n,j)$ if and only if $i \leq j$ and $(n,i) \leq (n,\omega)$, and $1 \leq (n,m)$ for all $(n,m) \in \omega \times (\omega+1)$.

Then, for each (n,ω) we build the following clopen local base. Let

$$A_{0,(n,\omega)} = \{-n, 0\} \cup (1, (n,\omega)]$$

and

$$A_{m,(n,\omega)} = ((n, m+1), (n,\omega)].$$

This example seems contrived, but witness the following terrible consequence. We have that $0 \in A_{0,(n,\omega)}$ for every n. But yet, $A_{m,(n,\omega)}$ is a clopen local base, and behaves quite well as expected. But, there is no way to separate each $A_{0,(n,\omega)}$ from $\{0\}$ in the sense that no finite set can successfully detect all neighborhoods which include 0.

It should be noted here that T is a Hausdorff, immediately branching, pseudo-suprema valent pseudotree.

4. CLASSIFICATION FOR SUPER-GENERATORS

In this section, we combine the previous discussions of topology to classify when immediately branching pseudo-suprema valent pseudotrees admit a discrete $(0,1)$-generator such that the range of each function in $\mathcal{G}$ is the two-point set $\{0,1\}$.

4.1. **Generators.**

First, we recall basic definitions from $C_p(X)$ theory about generators.

Definition 4.1. Let X be a Tychonoff space. Then we say that a collection of real-valued continuous functions on X, $\mathcal{G}$, is a generator for X if for every $x \in X$ and every non-empty closed set $K \subset X$ such that $x \notin K$, there exists a $g \in \mathcal{G}$ such that $g(x) \notin \overline{g(K)}$.

Definition 4.2. Let $C_p(X)$ be all continuous real-valued functions on a Tychonoff space X. We endow $C_p(X)$ with the point-wise topology. The basic open neighborhoods of any $f \in C_p(X)$ are given by, for all $\epsilon > 0$ and all $F \in [X]^{<\omega}$,

$$U(f, F, \epsilon) = \{g \in C_p(X) : \forall x \in F, |f(x) - g(x)| < \epsilon\}.$$

Definition 4.3. A generator $\mathcal{G}$ is *discrete* if $\mathcal{G}$ is a discrete subspace of $C_p(X)$.

Definition 4.4. A generator $\mathcal{G}$ is said to be a $(0,1)$-*generator* if for every $x \in X$ and non-empty closed set $K \subset X$ such that $x \notin K$, there is a $g \in G$ such that $g(x) = 1$ and $g(K) = \{0\}$.

Definition 4.5. A generator $\mathcal{G}$ is said to be *super* if it is a discrete $(0,1)$-generator with the property that for every $g \in \mathcal{G}$, $\operatorname{ran}(g) = \{0,1\}$.

We prove a short helpful proposition before going further.

Proposition 4.6. *Every continuous real-valued function on a space X such that the range is $\{0,1\}$ is of the form χ_A for some clopen $A \subset X$.*

Proof. Let $f : X \to \mathbb{R}$ be continuous with $\operatorname{ran}(f) = \{0,1\}$. For a fixed $0 < \epsilon < \frac{1}{2}$ and ball $B_\epsilon(1) \subset \mathbb{R}$, let $A = f^{-1}(B_\epsilon(1))$. Then, by continuity, A must be open. But, as f only takes values in $\{0,1\}$, it must also be the case that $A = f^{-1}(\{1\})$. Hence, A is also closed. Thus, $f(x) = 1$ if and only if $x \in A$, and $f(x) = 0$ if and only if $x \notin A$, and $f = \chi_A$. $\square$

4.2. Main Theorem.

We finally arrive at the main result.

Theorem 4.7. *Let T be a Hausdorff, immediately branching, pseudo-suprema valent, pseudotree equipped with the Sorgenfrey topology. Then the following are equivalent:*

i. *There exists a homogeneous pair $(\mathcal{M}, \mathcal{A})$ such that for every $t \in T$, $\mathcal{A}$ is agreeing above t, and if $\mathcal{M}$ is not finitely interacting with t, then $\mathcal{A}$ is agreeing below t.*
ii. *T admits a super generator.*

In short, this theorem says that if a pseudotree T admits a pair consisting of a uniform limiting collection of sequences converging to all limit points and an associated collection of clopen local bases such that the bases are monotone, and have the property that finite sets are suitable enough to detect all the neighborhoods that include a point, then we will obtain a super generator. The necessity result states that every super generator gives rise to this pair.

4.3. **Sufficiency.** We will break this proof up into multiple lemmas. First, sufficiency will be dealt with by showing that any homogeneous pair $(\mathcal{M}, \mathcal{A})$ will give rise to continuous real-valued functions by being the indicator on local basic sets. Then, we will show that the two hypotheses in *i.* are enough to ensure that these functions will be isolated in $C_p(T)$ by choosing a suitable finite set $F \in [T]^{<\omega}$.

Lemma 4.8. *Let $(\mathcal{M}, \mathcal{A})$ be a homogeneous pair such that for every $t \in T$, $\mathcal{A}$ is agreeing above t, and if $\mathcal{M}$ is not finitely interacting with t, then $\mathcal{A}$ is agreeing below t. Let*

$$\mathcal{G} = \bigcup_{\mathcal{A}_t \in \mathcal{A}} \{g_{\mu,t} : A_{\mu,t} \in \mathcal{A}_t\}.$$

where $g_{\mu,t} = \chi_{A_{\mu,t}}$. Then $\mathcal{G}$ is a $(0,1)$-generator.

Proof. As T is zero-dimensional, $(s,t]$ are clopen sets. Thus the functions $\chi_{(s,t]}$ are continuous for each $s < t$. Assume we have a point $t \in T$ and a closed subset K such that $t \notin K$. Then, as each $\mathcal{A}_t \in \mathcal{A}$ is a local basis for t, there exists some $\mu < \mathrm{cf}_T(t)$ and some $A_{\mu,t} \in \mathcal{A}_t$ such that $A_{\mu,t} \cap K = \emptyset$. But then, $g_{\mu,t}(t) = 1$ and $g_{\mu,t}(K) = \{0\}$ as desired. $\square$

Throughout, let $F^{\mathrm{a}}_{\mu,t}$ and $F^{\mathrm{b}}_{\mu,t}$ be the sets guaranteed by $\mathcal{A}$ agreeing above and below a point t and $\mu < \mathrm{cf}_T(t)$ respectively.

Lemma 4.9. *If t is isolated in T, then $g_{0,t}$ is isolated in $\mathcal{G}$.*

Proof. We will build a neighborhood of $g_{0,t}$ that misses all other $g_{\nu,t'} \in \mathcal{G}$. We have two possible cases: either $\mathcal{M}$ is finitely interacting with t or otherwise. In these two cases, we will pick a suitable finite set F. Let $U_{g_{0,t}} = U(g_{0,t}, F, \frac{1}{2})$.

Case 1. Assume $|\mathcal{M}(t)| < \omega$, and let $F = \{t\} \cup F^{\mathrm{a}}_{0,t} \cup \mathcal{M}(t)$.

If $t' \neq t$ is also isolated, then $|g_{0,t}(t) - g_{0,t'}(t)| = 1$. So, assume that t' is a limit. If $g_{\nu,t'}(t) = 0$, then $|g_{0,t}(t) - g_{\nu,t'}(t)| = 1$ as well. So, assume that $g_{\nu,t'}(t) = 1$. If $\phi_{t'}$ interacts with t, then there exists $s \in \mathcal{M}(t)$ such that $|g_{0,t}(s) - g_{\nu,t'}(s)| = 1$. If $\phi_{t'}$ does not interact with t, then $t \in A_{\nu,t'}$ by the fact that $g_{\nu,t'}(t) = 1$. So, there is $f \in F^{\mathrm{a}}_{0,t}$ such that $f \in A_{0,t} \Delta A_{\nu,t'}$. Hence, $|g_{0,t}(f) - g_{\nu,t'}(f)| = 1$.

Case 2. Assume $|\mathcal{M}(t)| \geq \omega$, and let $F = \{t\} \cup F^{\mathrm{a}}_{0,t} \cup F^{\mathrm{b}}_{0,t}$.

If $t' \neq t$ is also isolated or t' is a limit with $g_{\nu,t'}(t) = 0$, then it is the same as the previous case. So, assume t' is a limit with $g_{\nu,t'}(t) = 1$, then we have two cases. If $\phi_{t'}$ interacts with t, then there is an $f \in F^{\mathrm{b}}_{0,t}$ or if $\phi_{t'}$ does not interact with t, then there is an $f \in F^{\mathrm{a}}_{0,t}$ such that $f \in A_{0,t} \Delta A_{\nu,t'}$. Thus, $|g_{0,t}(f) - g_{\nu,t'}(f)| = 1$.

In both possible cases above, we have shown that $U_{g_{0,t}} \cap \mathcal{G} = \{g_{0,t}\}$ as every other function will differ from $g_{0,t}$ by 1 for some point in F. Hence, $g_{0,t}$ is isolated. □

Lemma 4.10. *If t is a limit in T, then $g_{\mu,t}$ is isolated in $\mathcal{G}$ for each $\mu < \mathrm{cf}_T(t)$.*

Proof. The strategy here is the same as in the last lemma. We fix a $\mu < \mathrm{cf}_T(t)$ and a $g_{\mu,t}$, and we construct a neighborhood that misses all other $g_{\nu,t'}$ by choosing an appropriate finite set F in the two possible cases. Let $U_{g_{\mu,t}} = U(g_{\mu,t}, F, \frac{1}{2})$.

Case 1. Assume $|\mathcal{M}(t)| < \omega$. Let $F = \{\phi_t(\mu), \phi_t(\mu+1), t\} \cup F^{\mathrm{a}}_{\mu,t} \cup \mathcal{M}(t)$. If t' is isolated, then $g_{0,t'}(t) = 0$, and $|g_{\mu,t}(t) - g_{0,t'}(t)| = 1$. So, assume now t' is also a limit. If $g_{\nu,t'}(t) = 0$, then we also have $|g_{\mu,t}(t) - g_{\nu,t'}(t)| = 1$.

So, assume that $g_{\nu,t'}(t) = 1$. If $t' \neq t$, then $t' > t$. If $\phi_{t'}$ interacts with t, then there is an $s \in \mathcal{M}(t)$ where $|g_{\mu,t}(s) - g_{\nu,t'}(s)| = 1$. If $\phi_{t'}$ does not interact with t, then there is $f \in F^{\mathrm{a}}_{\mu,t}$ such that $f \in A_{\mu,t} \Delta A_{\nu,t'}$. Then, $|g_{0,t}(f) - g_{\nu,t'}(f)| = 1$.

Now assume $t = t'$. Then it is the case that either $\phi_t(\mu) \in A_{\mu,t} \Delta A_{\nu,t'}$ or $\phi_t(\mu+1) \in A_{\mu,t} \Delta A_{\nu,t'}$ by the monotonicity of $\mathcal{A}$. Thus, we will get that $|g_{\mu,t}(\phi_t(\mu)) - g_{\nu,t'}(\phi_t(\mu))| = 1$ or $|g_{\mu,t}(\phi_t(\mu+1)) - g_{\nu,t'}(\phi_t(\mu+1))| = 1$.

Case 2. Assume $|\mathcal{M}(t)| \geq \omega$. Let $F = \{\phi_t(\mu), \phi_t(\mu+1), t\} \cup F^{\mathrm{a}}_{\mu,t} \cup F^{\mathrm{b}}_{\mu,t}$. If t' is isolated or if t' is a limit with $t \notin A_{\nu,t'}$, then $|g_{\mu,t}(t) - g_{\nu,t'}(t)| = 1$.

So, assume that t' is also a limit with $g_{\nu,t'}(t) = 1$. If $t' \neq t$, then $t' > t$. If $\phi_{t'}$ interacts with t, then there is an $f \in F^{\mathrm{b}}_{\mu,t}$ or if $\phi_{t'}$ does not interact with t, there is an $f \in F^{\mathrm{a}}_{\mu,t}$ where $f \in A_{\mu,t} \Delta A_{\nu,t'}$. So, in either case, we obtain that $|g_{\mu,t}(f) - g_{\nu,t'}(f)| = 1$.

Then, lastly, if $t = t'$, the case is the same as above. We apply monotonicity of $\mathcal{A}$ to deduce that either $\phi_t(\mu) \in A_{\mu,t} \Delta A_{\nu,t'}$ or $\phi_t(\mu+1) \in A_{\mu,t} \Delta A_{\nu,t'}$ to obtain that either $|g_{\mu,t}(\phi_t(\mu)) - g_{\nu,t'}(\phi_t(\mu))| = 1$ or $|g_{\mu,t}(\phi_t(\mu+1)) - g_{\nu,t'}(\phi_t(\mu+1))| = 1$.

In both cases, we have exhibited an F such that $U(g_{\mu,t}, F, \frac{1}{2}) \cap \mathcal{G} = \{g_{\mu,t}\}$ as every other function different from $g_{\mu,t}$ will differ by at least 1 on the finite set F. Hence $g_{\mu,t}$ is isolated in $\mathcal{G}$. □

Lemma 4.11. *$\mathcal{G}$ is a super generator.*

Proof. By lemma 4.8, $\mathcal{G}$ is a $(0,1)$-generator, and lemmas 4.9 and 4.10 show that $\mathcal{G}$ is discrete. □

4.4. **Necessity.**

Now we prove necessity. Necessity will be established in a few steps. First, we will show that every super generator $\mathcal{G}$ determines an associated homogeneous pair. Then, it will be shown that $\mathcal{G}$ will actually have a canonical sub-generator $\mathcal{H}$ made up of indicator functions whose sets are from the collection of clopen local bases. These indicator functions will be enough to ensure the properties needed of the homogeneous pair $(\mathcal{M}, \mathcal{A})$. So, assume $\mathcal{G}$ is a super generator on a Hausdorff, immediately branching, pseudo-suprema valent pseudotree T.

Lemma 4.12. *$\mathcal{G}$ determines a homogeneous pair $(\mathcal{M}, \mathcal{A})$.*

Proof. We will construct a cofinal sequence t_μ for every limit point t from clopen subsets $A_{\mu,t}$ determined by the inverse image of a function in $\mathcal{G}$ which is sufficiently separating. Both of these are enough to guarantee limiting maps, monotoncity, and homogeneity.

Fix a limit point t and any $s < t$ with $\mathrm{cf}_T(t) = \mathrm{cf}(h_s(t))$ and a cofinal sequence $\{t_\mu : \mu < \mathrm{cf}_T(t)\}$ with $t_0 \geq s$. Moreover, fix some $1 > \epsilon > 0$, and a neighborhood $D = B_\epsilon(1) \subset \mathbb{R}$.

For $\mu = 0$, there exists a function $g_{0,t}$ such that $g_{0,t}$ separates t from $(t_0, t]^c$. Set $A_{0,t} = g_{0,t}^{-1}(D)$. Then by continuity, $t \in A_{0,t} \subset (t_0, t]$ is open. Hence, there exists a $t_0 \leq s_0 < t$, such that $s_0 < t$ and $(s_0, t] \subset A_{0,t}$. Indeed, may pick the smallest $(s_0, t]$. To find s_0, use Dedekind completeness. Consider the chain $C = \{s < t : (s, t] \not\subset A_{0,t}$ It must be bounded and non-empty as $A_{0,t}$ is open. Then, allow s_0 to be the least upper bound. Then, let $\phi_t(0) = s_0$.

Assume we have defined s_μ and $A_{\mu,t}$. Then, there exists a $g_{\mu+1,t} \in \mathcal{G}$ such that $g_{\mu+1,t}$ separates t and $(\max\{s_\mu, t_\mu\}, t]^c$. Set $A_{\mu+1,t} = g_{\mu+1,t}^{-1}(D)$. Then, as $g_{\mu+1,t}$ is continuous, and $A_{\mu+1,t}$ is open, there exists $t_\mu < s_{\mu+1} < t$ such that $(s_{\mu+1}, t] \subset A_{\mu+1,t}$. Again, we may take $s_{\mu+1}$ to be the smallest, and let $\phi_t(\mu + 1) = s_{\mu+1}$

Assume we have defined s_μ for all $\mu < \lambda < \mathrm{cf}_T(t)$ where λ is a limit. The set $C = \{s_\mu : \mu < \lambda\}$ is a chain. Then, by Dedekind completeness, there exists a least upper bound c_λ. Note, that $c_\lambda \neq t$ as it would contradict cofinality. Then, there must exist a $g_{\lambda,t}$ that separates t from $(\max\{c_{\lambda\}, t_\lambda}, t]^c$. Set $A_{\lambda,t} = g_{\lambda,t}^{-1}(D)$. Then, by continuity, $A_{\lambda,t}$ is open. So, there exists an $t_\lambda < s_\lambda < t$ such that $(s_\lambda, t] \subset A_{\lambda_t}$. Set $\phi_t(\lambda) = s_\lambda$.

Hence, we have defined a sequence $\{s_\mu : \mu < \mathrm{cf}_T(t)\}$ and a collection of local bases of clopen sets $\mathcal{A} = \{A_{\mu,t} : \mu < \mathrm{cf}_T(t)\}$.

Then, we claim that $\phi_t(\mu) = s_\mu$ is limiting and $\mathcal{A}$ is a monotone local basis. But ϕ_t being limit is immediate. s_μ was constructed in such that $s_\mu \geq t_\mu$ and $\{s_\mu\}$ is monotonic. Moreover, it must be the case $s_\mu \to t$ as $t_\mu \to t$. Moreover, $\mathcal{A}$ automatically becomes a local basis. If $(s, t]$ is a basic open neighborhood, then there exists a $s_\mu > s$ by cofinality, and $A_{\mu+1} \subset (s_\mu, t] \subset (s, t]$. The homogeneity condition is built into the constructio nas well by the previous statement and definition of $\phi_t(\mu)$. $\square$

Proposition 4.13. *For each isolated $t \in T$, the function $g_t = \chi_{\{t\}} \in \mathcal{G}$.*

Proof. If t is isolated, then $\{t\}$, by the generator property, there is a $g \in \mathcal{G}$ such that $g(t) = 1$ and $g(\{t\}^c) = \{0\}$. Hence, $g = g_t = \chi_{\{t\}}$. □

Lemma 4.14. *Let $\{A_{\mu,t} : \mu < \mathrm{cf}_T(t)\}$ be a local basis at a limit point t guaranteed by $\mathcal{G}$. Define*

$$\mathcal{H} = \left(\bigcup_{t \in T \setminus T'} \{\chi_{\{t\}}\} \right) \cup \left(\bigcup_{t \in T'} \{\chi_{A_{\mu,t}} : \mu < \mathrm{cf}_T(t)\} \right).$$

Then $\mathcal{H} \subset \mathcal{G}$ and $\mathcal{H}$ is a discrete super $(0,1)$-generator.

Proof. This is immediate by construction of $A_{\mu,t}$ from $\mathcal{G}$ and proposition 4.13. □

Lemma 4.15. *If $(\mathcal{M}, \mathcal{A})$ is the homogeneous pair determined by $\mathcal{G}$, then $\mathcal{A}$ is agreeing above all $t \in T$, and if $\mathcal{M}$ is not finitely interacting with t, then $\mathcal{A}$ is agreeing below t.*

Proof. By way of contradiction, assume that $\mathcal{A}$ is not agreeing above at every $t \in T$. Then there exists some point t and $\mu < \mathrm{cf}_T(t)$, such that for every $F \in [T]^{<\omega}$, we can find another point t' and $\nu < \mathrm{cf}_T(t')$ such that $(A_{\mu,t} \Delta A_{\nu,t'}) \cap F = \emptyset$. Then, there is no way that $\mathcal{G}$ can be discrete, as this shows that every neighborhood $U(g_{\mu,t}, F, \epsilon)$ will also contain some $g_{\nu,t'}$.

Assume $\mathcal{M}$ is not finitely interacting with t and $\mathcal{A}$ is not agreeing below t. Then we have $\lambda_t \geq \omega$, and the set of limits above $t' > t$ such that $\phi_{t'}$ interacts with t is not empty. Thus, if $F \in [T]^{<\omega}$ is arbitrary, then we are guaranteed a limit $t' > t$ with $\nu < \mathrm{cf}_T(t')$ with $|g_{\mu,t}(f) - g_{\nu,t'}(f)| = 0$ for all $f \in F$. Hence, every neighborhood of $g_{\mu,t}$ will also contain another $g_{\nu,t'}$, and $\mathcal{G}$ cannot be discrete. □

Thus, sufficiency of theorem 4.7 is covered by lemma 4.11, and necessity is proven by lemma 4.15

4.5. **Immediate Corollaries and Possible Directions.** We end this paper with by stating a few immediate corollaries as they relate to trees and ordinals.

Proposition 4.16. *Let T be a tree. Then there exists a homogeneous pair $(\mathcal{M}, \mathcal{A})$ such that $\mathcal{A}$ is agreeing above every $t \in T$.*

Proof. For each $t \in T$, if t is the root or of successor height, then $A_{0,t} = \{t\}$ and ϕ_t is defined as in definition 3.22. Let t be of limit height. Choose any strictly monotone cofinal sequence $\{t_\mu : \mu < \mathrm{cf}(h(t))\}$. Then let $A_{\mu,t} = (t_\mu, t]$. Then, simply let $\phi_t(\mu) = t_\mu$. If $\mathcal{M}$ is all of these limit maps and $\mathcal{A}$ is all the local bases above, then it is immediate that $\mathcal{A}$ is monotone and the pair is homogeneous.

To see that $\mathcal{A}$ is finitely agreeing, we simply note the following: if $A_{\mu,t}$ is a neighborhood of t and $A_{\nu,t'}$ is a neighborhood of t', then $t \in A_{\nu,t'}$ if and only if $\phi_{t'}(\nu) < t'$. Thus, agreeing above is vacuously satisfied. $\square$

Corollary 4.17. *Every finite branching Hausdorff tree will have a super generator.*

Proof. Finite branching Hausdorff trees are automatically immediately branching and pseudo-suprema valent. Moreover, $\lambda_t < \omega$ for all $t \in T$. Thus, $\mathcal{M}$ is finitely interacting at every point. $\square$

Corollary 4.18. *There exist trees of arbitrary height and breadth that exhibit a super generator.*

Proof. Fix two cardinals κ and μ. Then, consider $\kappa + 1$ as a tree and add on to the root with μ many immediate successors. $\mathcal{M}$ will be finitely interacting with every point. Hence, apply proposition 4.15. $\square$

Corollary 4.19. *Every ordinal exhibits a super generator.*

Proof. Apply proposition 4.16 as ordinals are trees where $\lambda_t \leq 1$ for every $t \in \alpha$. $\square$

Lastly, we end with two possible possible continuations of research.

Problem 4.20. Is such a classification available for Hausdorff, non-immediately branching, pseudo-suprema valent pseudotrees?

Problem 4.21. What properties of trees gives rise to an equivalence of the existence of such a homogeneous pair $(\mathcal{M}, \mathcal{A})$ with the properties that admit super generators?

References

[1] Lynne Baur. Cardinal functions on initial chain algebras on pseudotrees. *Order*, 17(1):1–21, Mar 2000.

[2] M. Bekkali. Pseudo treealgebras. *Notre Dame Journal of Formal Logic*, 42(2):101–108, 2001.

[3] Ryszard Engelking. *General Topology*, volume Vol. 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, revised and completed edition edition, 1989.

[4] Z. Feng, P. Gartside, and A. Yuschik. Discrete generators. *Acta Mathematica Hungarica*, 166(1):162–178, Feb 2022.

[5] Thomas Jech. *Set Theory*. Springer, 2002.

[6] Sabine Koppelberg and J. Donald Monk. Pseudo-trees and boolean algebras. *Order*, 8(4):359–374, 1991.

[7] Stephen Willard. *General Topology*. Dover, 2004.

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