An Introduction to Forcing

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Table of Contents

Purpose of this Talk

Forcing was first invented by Paul Cohen in his proof of the independence of the Continuum Hypothesis from the usual axioms of set theory. The goal of this talk today is give an introduction to the method of forcing, while covering the necessary preliminaries, and, also, give a proof of the independence of CH.

The Axioms of ZFC and Independence

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ZFC

We won't give a statement of every axiom or axiom schema in ZFC, but they include things that we as mathematicians use every day.

1. Axiom of Extensionality: two sets x and y are equal if they contain the same elements.
2. Axiom schema of Replacement: the image of any set under a function will be a set.
3. Axiom of Powerset: for any set x, there exists a set consisting of all subsets of x.

Independence and Consistency

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What we want to show is that the Continuum Hypothesis is, in fact, independent of ZFC. To do this, it would be enough to show that both CH and $\neg$CH are both consistent.

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Take $\mathcal{T}$ to be theory of groups. Its axioms are the familiar ones. Then a model of $\mathcal{T}$ will simply be a group. For a concrete example, $S = \{z \in \mathbb{C} : |z| = 1\}$ is a group where the group operation is multiplication. So S is a model of group theory.

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Models are quite nice, but quite difficult to get our hands on. In particular, in ZFC, it is only consistent that a countable model of ZFC exists - not even provable. However, the consistency of ZFC implies there is indeed a countable one.

Models (cont.)

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This property will be incredibly useful. It guarantees that ordinals in our model M are real ordinals. They are not subsets that look like ordinals, but are indeed ordinals that appear in our universe V. Moreover, it means that M will believe it has all the ordinals of V even though it has countably many of them.

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Fact 1

Models of ZFC are indeed sets inside the universe V. They satisfy the axioms of ZFC *relatively*. Meaning, that the axioms are relativized to the set M - all quantifiers are restricted to being in M. So, every time you see a $\forall x$, restrict it to $\forall x \in M$ and similarly for $\exists x$ gets restricted to $\exists x \in M$.

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A countable transitive model M does not really contain all the cardinals of the universe V. Otherwise, it wouldn't be countable. But it contains a relativized version of them. How?

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Important Metatheorems

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Suppose ϕ is a sentence of set theory. If we can find a model of ZFC that satisfies ϕ in ZFC$+M$, then, if ZFC is consistent then so is ZFC $+ \phi$.

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Metatheorem 2

Suppose ϕ is a sentence of set theory. If we can find a model of ZFC that satisfies ϕ in ZFC+M, then, if ZFC is consistent then so is ZFC + ϕ.

So, the goal of finding independence results is to construct models that satisfy ϕ and $\neg\phi$, so we will know that ϕ and $\neg\phi$ will be consistent with ZFC, hence implying ϕ is independent.

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This example highlights a problem that we will tackle. Given a countable transitive model M of ZFC and some sentence ϕ in the language of set theory, can we find the smallest extension of M that remains a model of set theory and satisfies the sentence ϕ? The answer to this will be the process of forcing.

Forcing Notions

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Ideals

An *ideal* of a forcing notion $\mathbb{P}$ is a subset $G \subset \mathbb{P}$ such that:

i. if $q \in G$ and $p \leq q$, then $p \in G$
ii. if $p_1, p_2 \in G$ then there exists a common extension also lying in G .

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Example

We let $\mathbb{P}$ be the set of partial functions from ω to $\{0, 1\}$. We order $\mathbb{P}$ by whether a function extends another, i.e. $f \leq g$ if and only if $\operatorname{dom}(f) \subset \operatorname{dom}(g)$ and $f(n) = g(n)$ for all $n \in \operatorname{dom}(f)$.

The sets $D_k = \{f : k \in \operatorname{dom}(f)\}$ is a dense in $\mathbb{P}$, so any generic over $\mathbb{P}$ will represent a full function from ω to $\{0, 1\}$. Why? Well, G will have a function f_k that has k in its domain. Moreover, any two functions in G must be compatible. So, we can piece together a single function from the compatible ones found in G.

The Generic Extension $M[G]$

Sadly, we do not get to witness the full construction of $M[G]$ as it was intended due to time constraints. The only real important thing I can tell you is that $M[G]$ will be the smallest model that extends M and contains our generic G. This can be summarized in the following facts.

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The proof of the above fact is quite technical. It involves introducing the Fundamental Theorem of Forcing which, essentially, states that the existence of certain conditions p in the generic G from $\mathbb{P}$ will make $M[G]$ satisfy certain sentences.

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We have $M \subset M[G]$.

This can be done easily as for every $x \in M$, we can define a $\mathbb{P}$-name that evaluates to x. Let $\check{x} = \{(\check{y}, p) : y \in x, p \in \mathbb{P}\}$. When evaluated, $\check{x}^G = x$.

Forcing the Negation of the Continuum Hypothesis

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Continuum Hypothesis (CH)

If $\mathbb{N} \subset S \subset \mathbb{R}$, then $|S| = |\mathbb{R}|$ or $|S| = |\mathbb{N}|$. In words, it says that there is no subset of the real numbers that has cardinality strictly between the cardinality of the reals or the cardinality of the naturals. In terms of cardinals, CH is written as

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So, our goal now reduces to the following: lets start with a countable transitive model M, and expand it to a model of ZFC such that $\neg$CH fails. By the above metatheorems, this will show that ZFC + $\neg$CH is consistent.

Forcing ¬CH

We will now begin to describe the forcing notions $\mathbb{P}$ that will assist us in showing ¬CH holds. The idea to make this happen is to introduce $\aleph_2$ many subsets of $\mathbb{N}$. This way, it guarantees that $2^{\aleph_0} > \aleph_1$.

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Our Forcing Notion

We will let $\mathbb{P}$ be the set of partial functions from $\aleph_0 \times \aleph_2$ onto $\{0, 1\}$ ordered by extension, i.e. $f \leq g$ if and only if $\mathrm{dom}(f) \subset \mathrm{dom}(g)$ and $g = f$ for all values on $\mathrm{dom}(f)$.

Forcing $\neg$CH

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Simple enough right? Now let's see why a generic ideal of $\mathbb{P}$ introduces $\aleph_2$ many subsets of $\mathbb{N}$.

Forcing ¬CH

Generic ideal for $\mathbb{P}$

Any generic G of $\mathbb{P}$ will represent a full function from $\aleph_2 \times \aleph_0$ to $\{0, 1\}$. This means, that for every $\alpha < \aleph_2$, we get a full function $f_\alpha : \aleph_0 \to \{0, 1\}$. Hence, we get $\aleph_2$ many subsets of $\aleph_0$.

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Are we done, then? No. Let me introduce two problems. How do we know that, when we expanded M to $M[G]$, that we didn't introduce a new set that is a bijection from $\aleph_1$ to $\aleph_2$? This process is called cardinal collapse and is useful for some forcing notions. However, that would be devastating. Also, how do we know these subsets introduced are all distinct? What if they're not, then we can certainly have less than $\aleph_2$ many subsets introduced which is also devastating. We will remedy these two questions.

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The name c.c.c. is horrendous given what property it describes, but it is a name that has stuck.

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Theorem

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Theorem

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Proof

Let G be a generic over $\mathbb{P}$, and let $\alpha \in M$ be an infinite ordinal that is not a cardinal in $M[G]$. Then, there exists an infinite ordinal $\beta \in M$ such that there exists a surjection $f : \beta \to \alpha$ in $M[G]$. Then there is a function $g : \beta \to \mathcal{P}(\alpha)$ such that $f(\gamma) \in g(\gamma)$ and $g(\gamma)$ is countable in M. Then, $\alpha = \bigcup_{\gamma \in \beta} g(\gamma)$. Hence, $|\alpha| = |\bigcup_{\gamma \in \beta} g(\gamma)| = |\beta| \cdot \aleph_0 = |\beta|$. So α is not a cardinal in M.

Our new subsets are distinct

Now, we know that $M[G]$ has the exact same cardinals as M. So, we haven't introduced anything that would mess up the fact that we are really introducing $\aleph_2$ many subsets f_α. However, we still need to check that these subsets f_α are, in fact, distinct.

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Proof

Let f be the full function created by our generic G. For each ordinal $\alpha < \aleph_2$, define the sets

$$f_\alpha^{-1}(1) = \{n \in \aleph_0 : f(n, \alpha) = 1\}$$

Then for any distinct ordinals $\alpha, \beta < \aleph_2$, the set $D_{\alpha,\beta} = \{g \in \mathbb{P} : \exists n((n, \alpha), (n, \beta) \in \mathrm{dom}(g)$ and $g(n, \alpha) \neq g(n, \beta))\}$ is dense in $\mathbb{P}$. So, G must intersect each of these D's. This guarantees that f_α^{-1} are distinct sets which were created by f.

Sources

1. Nik Weaver's *Forcing for Mathematicians*
2. Thomas Jech's *Set Theory*
3. Joseph Shoenfield's *Mathematical Logic*