Countably Generated Complete Boolean Algebras of Arbitrary Size

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Main Result

Theorem (Solovay, 1966)

For every ordinal λ, there exists a countably generated complete Boolean algebra of size larger than $\aleph_\lambda$.

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Theorem (Solovay, 1966)

For every ordinal λ, there exists a countably generated complete Boolean algebra of size larger than $\aleph_\lambda$.

Solovay's original proof is rather unintuitive. Moreover, his result can be tightened to show that those Boolean algebras are of size exactly $2^{\aleph_\lambda}$. The key ingredient is some elementary properties of the Baire Space of Weight λ.

Orderings

Definition

A set W with binary relation $<$ is *linearly ordered* if

i. $p \not< p$
ii. if $p < q$ and $q < r$, then $p < r$
iii. and either $p < q$, $q < p$ or $p = q$

for all $p, q, r \in W$.

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Definition

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Example

The natural numbers, $\mathbb{N}$, is a well-ordered set. Every subset of a well ordered set is well-ordered.

Ordinals

Informally, ordinals are a special type of well-ordered set such that the entire class of ordinals is well-ordered by $\subset$ relation (however, it is usually still written as $<$). Moreover, every-well ordered set must be order isomorphic to some ordinal.

Ordinal Conventions

Ordinals are usually written as lowercase Greek letters, α, β, γ, etc. except in the finite case.

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The *finite ordinals* are represented by natural numbers n, and they correspond to the well-ordered set whose cardinality in equal to n. Actually, each n can be represented as the well-ordered set $\{m \in \mathbb{N} : m < n\}$.

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For any given ordinal λ, the ordinal ω_λ corresponds to the first infinite ordinal that has cardinality $\aleph_\lambda$. Here, $\aleph_\lambda$ is the first cardinality that is strictly larger than $\aleph_\alpha$ for all $\alpha < \lambda$.

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Example

The ordinal $\omega_0 = \omega$ corresponds to natural numbers and has size $\aleph_0 = |\mathbb{N}|$. The ordinal ω_1 corresponds to the first uncountable well ordered set.

Boolean algebras

Definition

A *Boolean algebra* is a nonempty set $\mathscr{B}$ with two distinguished elements 0 and 1, two binary operators $\vee$ and $\wedge$, and unary operation ^{c} which satisfy the following axioms:

$$p \vee q = q \vee p \qquad p \wedge q = q \wedge p \tag{1}$$

$$p \vee (q \vee r) = (p \vee q) \vee r \qquad p \wedge (q \wedge r) = (p \wedge q) \wedge r \tag{2}$$

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \qquad p \vee (q \wedge r) = (p \vee q) \wedge (p \wedge r) \tag{3}$$

$$p \wedge (p \vee q) = p \qquad p \vee (p \wedge q) = p \tag{4}$$

$$p \vee p^c = 1 \qquad p \wedge p^c = 0 \tag{5}$$

for every $p, q, r \in \mathscr{B}$.

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for every $p, q, r \in \mathscr{B}$.

From the above axioms, we can conclude the following basic results:

$$p \vee 1 = 1 \qquad p \wedge 1 = p$$

$$p \vee 0 = p \qquad p \wedge 0 = 0$$

$$0^c = 1 \qquad 1^c = 0$$

Examples of Boolean algebras

Example

The trivial Boolean algebra only contains 0 and 1. Meet, join, and complement behave exactly like logical conjunction, disjunction, and negation, respectively.

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Let X be any set. Then the powerset, $\mathscr{P}(X)$, is a Boolean algebra under the operations of union, intersection, and complementation. The 0 and 1 elements are identified to $\emptyset$ and X respectively.

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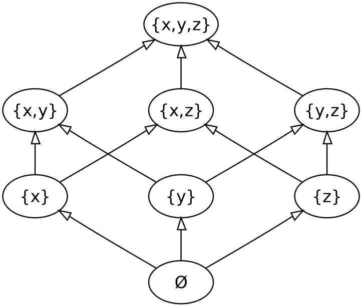

Generators of Boolean Algebras

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Definition

A Boolean algebra is *countably generated* if it contains a finite or denumerable generator.

Complete Boolean Algebras

Every Boolean algebra $\mathscr{B}$ is actually a partially ordered set under the following ordering:

$$p \leq q \qquad \text{if and only if} \qquad p \vee q = q \qquad \text{if and only if} \qquad p \wedge q = p$$

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for any $p, q \in \mathscr{B}$.

Definition

Let S be a subset of a Boolean algebra $\mathscr{B}$. Then we can define the *arbitrary join and meet of* S as

i.
$$\bigvee\{s \in S\} = \bigvee S = \sup S$$

ii.
$$\bigwedge\{s \in S\} = \bigwedge S = \inf S$$

where $\inf$ and $\sup$ are taken over the partial ordering defined above.

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A word of caution: we are only defining $\bigvee S$ and $\bigwedge S$ when $\sup$ and $\inf$ actually exist.

Complete Boolean Algebras

We note that for any finite $S \subset \mathscr{B}$, $\bigvee S$ and $\bigwedge S$ align perfectly with the already defined meet and join of their elements.

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The most accessible examples of complete Boolean algebras are finite Boolean algebras or $\mathscr{P}(X)$ for any set X. They are complete by way of definition of the intersection and union.

Topologies

Definition

Let X be a set. A subset $\tau \subset \mathscr{P}(X)$ is a topology on X if

i. both $\emptyset$ and X are in τ
ii. finite intersections of elements in τ are in τ
iii. arbitrary unions of elements in τ are in τ

We declare the elements of τ to be open sets. Moreoever, a set V is closed if and only if V^c is open. A set V is clopen if it is both open and closed.

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Example

Let X be any set. If $\tau = \mathscr{P}(X)$, then X is said to have the discrete topology. If $\tau = \{\emptyset, X\}$, then X is said to have the indiscrete topology.

Generators for Topologies

Sometimes it's convenient to describe a topology in terms of generators instead of looking at what the entire collection of open sets look like.

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Definition

Given a topological space X, a *base* for X is a subcollection of open sets $\mathscr{B}$ such that

i. every open set is equal to the union of elements from the base
ii. if any two base elements B_1 and B_2 have non empty interseciton, then there is a base element B_3 such that $B_3 \subset B_1 \cap B_2$.

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Example

The base for the usual topology on $\mathbb{R}$ is given by the set $\mathscr{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$.

Interior and Closure

Let X be a topological space, and $A \subset X$.

Definition

The *interior* of A is the largest open set contained in A. It is given by

$$\operatorname{int}(A) = \bigcup \{U \subset A : U \text{ is open}\}$$

The *closure* of A is the smallest closed set containing A. It is given by

$$\operatorname{cl}(A) = \bigcap \{V \supset A : V \text{ is closed}\}$$

Regular Open Sets

Definition

Let X be a topological space. A subset A of X is called *regular open* when $\mathrm{int}(\mathrm{cl}(A)) = A$.

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A set that is open and closed is always regular open.

Regular Open Sets as Boolean Algebras

Theorem

Let X be a topological space, and let $\mathscr{B}$ be the collection of all regular open sets of X. Then $\mathscr{B}$ forms a complete Boolean algebra where $\emptyset$ is the 0 element, X is the 1 element, and for any $\mathscr{U} \subset \mathscr{B}$,

$$\bigvee \mathscr{U} = \operatorname{int}\left(\operatorname{cl}\left(\bigcup \mathscr{U}\right)\right)$$

for any $\mathscr{U} \subset \mathscr{B}$ with $\mathscr{U} \neq \emptyset$

$$\bigwedge \mathscr{U} = \operatorname{int}\left(\bigcap \mathscr{U}\right)$$

and for any $U \in \mathscr{B}$,

$$U^c = \operatorname{int}(X \setminus U)$$

Products

Definition

Let A be a set, and, for every $\alpha \in A$, let X_α be a set. Then we define the *Cartesian product* of X_α to be

$$\prod_{\alpha \in A} X_\alpha := \{f : A \to \bigcup_{\alpha \in A} X_\alpha : f(\alpha) \in X_\alpha\}$$

When all X_α are equal, we simply write the product as X^A.

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Example

Let X_1 and X_2 be $\mathbb{R}$. Then $\prod_{i \in \{0,1\}} X_i$ aligns with our previous notion of what $\mathbb{R}^2$ should be. Ordered pairs of the form (r_0, r_1) are precisely the set of functions from the two element set $\{0, 1\}$ such that $f(0) = r_0 \in \mathbb{R}$ and $f(1) = r_1 \in \mathbb{R}$.

Tychonoff Topology

We begin to ask ourselves, can we construct a natural enough topology on $\prod_{\alpha \in A} X_\alpha$ where each X_α is a topological space?

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Definition

Let A be a set, and, for each $\alpha \in A$, let X_α be a topological space. We define the *Tychonoff topology* on the set $\prod_{\alpha \in A} X_\alpha$ to be the topology whose base consists of the sets of the form

$$\prod_{\alpha \in A} U_\alpha$$

where

i. U_α is open in X_α for every α
ii. $U_\alpha = X_\alpha$ for all but finitely many α

The Baire Space of Weight λ

We start defining our space by fixing some ordinal λ

Definition

We define the *Baire Space of Weight* λ to be the product space

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For the rest of talk, let X be the Baire Space of Weight λ.

Description of X

What does X even look like?
From our previous definitions, any element $f \in X$ is a function $f : \omega \to \omega_\lambda$. It can be thought of as a sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ where $\lambda_i \in \omega_\lambda$ for all $i < \omega$.

Partitions of X

Let $f \in X$ and $n < \omega$. Then define

$$U(n, f) = \{g \in X : \forall m \leq n, g(m) = f(m)\}$$

We note that each $U(n, f)$ is actually a basic open set.

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Define the set $B_n = \{U(n, f) : f \in X\}$

Proposition

The set B_n partitions X into disjoint clopen sets.

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Proposition

The set B_n partitions X into disjoint clopen sets.

Proof sketch

Each $g \in X$ will certainly lie in some element of B_n, namely $U(n, g)$. If $U(n, g)$ and $U(n, f)$ have nonempty intersection, then f and g agree at all $i \leq n$, but then $U(n, f) = U(n, g)$. To show $U(n, f)$ is clopen, we show that it is open and its complement is as well.

Refinements of Partitions of X

Proposition

For $n < m$, we will have B_m refining B_n. Meaning, for all $U \in B_m$, there is some $V \in B_n$ such that $U \subset V$.

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Proposition

The collection $\{B_n\}_{n<\omega}$ forms a countable set of partitions of X into clopen sets such that each successive partition refines the last.

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Proposition

Every open set U of X can be written as a countable union of disjoint clopen sets.

Proof of Last Proposition

Proof

Define the function $\phi_U : U \to \omega$ as $\phi_U(f) = \inf(n < \omega : U(n, f) \subset U)$. This function is well defined defined by the previous proposition. Next, let $U_n = \bigcup\{U(n, f) : \phi_U(f) = n\}$. All U_n are pairwise disjoint. For any $f \in U$, if $f \in U_n$ and $f \in U_m$, then $n = \phi_U(f) = m$, which cannot happen unless $n = m$. It's also clear that each U_n is clopen as U_n is a union of basic open sets, and $X \setminus U_n$ is simply the union of the open elements of B_n that are not in U_n.

Finally, we see that

$$U = \bigcup_{n<\omega} U_n$$

For all $f \in U$, the set $\{n < \omega : U(n, f) \subset U\}$ is nonempty by the fact that $U(n, f)$ is a basic open set. Thus, $f \in U_n$ for some $n < \omega$, and $U \subset \bigcup_{n<\omega} U_n$. The reverse inclusion is clear as $U_n \subset U$ for all n.

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Let $\mathscr{B}$ be the Boolean algebra of regular open sets of the Baire Space of Weight λ.

We already know that the Boolean algebra of regular open sets is complete. We only need to show that it is countable generated and has size $2^{\aleph_\lambda}$.

Generators for $\mathscr{B}$

For every ordinal $\eta < \omega_\lambda$, define the set

$$A_{n,\eta} = \{f \in X : f(n) = \eta\}$$

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The collection $\{A_{n,\eta} : n < \omega, \eta < \omega_\lambda\}$ generates $\mathscr{B}$.

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Proposition

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Proof sketch

Given any regular open set U, it is also open. By the previous proposoitions $U = \bigcup_{i<\omega} U_i$ where U_i is clopen and pairwise disjoint. Then each

$$U(n, f) = \bigwedge_{i \leq n} A_{i,f(i)}$$

and

$$U_n = \bigvee \{U(n, f) : \phi_U(f) = n\}$$

and

$$U = \bigvee_{n<\omega} U_n$$

.

Countable Generators of $\mathscr{B}$

For every $n, m < \omega$, define the set $B_{n,m} = \{f \in X : f(n) \leq f(m)\}$.

Proposition

For every $n, m < \omega$, $B_{n,m} \in \mathscr{B}$.

Countable Generators of $\mathscr{B}$

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Proposition

For every $n, m < \omega$, $B_{n,m} \in \mathscr{B}$.

Proof sketch

It suffices to show $B_{n,m}$ is clopen. Write $B_{n,m}$ and $X \setminus B_{n,m} = \{f \in X : f(n) > f(m)\}$ as unions of basic clopen sets.
The idea is that $B_{n,m}$ is the union of sets whose nth element is an ordinal less than the mth element. We union over all possibilities.

Countable Generators of $\mathscr{B}$

Proposition

The Boolean algebra $\mathscr{B}$ is generated by all $B_{n,m}$.

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Proof setup

The proof is completed by transfinite induction. We first let $\mathscr{B}'$ to be the smallest complete Boolean algebra that contains $B_{n,m}$ for all $n, m < \omega$. Then, in order to show $\mathscr{B}' = \mathscr{B}$, we simply show that $A_{n,\eta} \in \mathscr{B}'$ for every $n < \omega, \eta < \omega_\lambda$. Then we will be done as $B_{n,m}$ must then generate $\mathscr{B}$.
We define the sets

$$C_{n,\eta} = \{f \in X : f(n) \leq \eta\}$$

and

$$Z_{n,\eta} = \{f \in X : f(n) < \eta\}$$

A crucial observation is that

$$A_{n,\eta} = C_{n,\eta} \wedge Z^c_{n,\eta}$$

So, if we show $C_{n,\eta}$ and $Z_{n,\eta}$ in $\mathscr{B}'$, we will be done.

Partial Proof of Last Proposition

Proof

We will only show part of the induction step. For any $n < \omega$, we assume we have shown $C_{n,\xi}, Z_{n,\xi} \in \mathscr{B}'$ for all $\xi < \eta$.
The first claim is that

$$Z_{n,\eta} = \bigvee_{\xi<\eta} A_{n,\xi}$$

Briefly, if $f \in Z_{n,\eta}$ then $f(n) < \eta$, so it's in some $A_{n,\xi}$. If f is in the right hand side, then clearly $f(n) = \xi < \eta$. So it's in the left.
The next step is to show that

$$C_{n,\eta} = \bigwedge_{m<\omega} \left(Z_{m,\eta} \bigcup B_{n,m}\right)$$

If f is in the right hand side, then $f(n) \leq \eta$. So, for each $m < \omega$ either $f(m) < \eta$ or $f(m) \geq \eta \geq f(n)$. In either case, f will be in $Z_{m,\eta}$ or $B_{n,m}$.

Proof (cont.)

Proof (cont.)

We show the reverse inclusion by contrapositive. Assume $f \notin C_{n,\eta}$. Then, consider the set $U(N, f)$ for some fixed $N > n$. Define the funciton

$$h_N(m) = \begin{cases} f(m) & m \leq N \\ \eta & m > N \end{cases}$$

The function h_N will lie in $U(N, f)$, but h_N is not contained in $Z_{N,\eta} \bigcup B_{n,N}$ for any $m < \omega$. Therefore f is not an interior point of

$$\bigcap_{m<\omega} \left(Z_{m,\eta} \bigcup B_{n,m} \right)$$

and it will not lie in the meet.

Cardinality of the Set of Open Sets of X

Proposition

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Proof

Every open set U is the countable union of disjoint clopen sets. Each one of those clopen sets is taken from a union of elements in the partition B_n. The the total number of combinations to choose from each B_n is precisely

$$(2^{\aleph_\lambda})^n = 2^{\aleph_\lambda}$$

Therefore, each open set may be written in at most

$$(2^{\aleph_\lambda})^{\aleph_0} = 2^{\aleph_\lambda \cdot \aleph_0} = 2^{\aleph_\lambda}$$

ways. Hence, the number of open sets is bounded above by $2^{\aleph_\lambda}$.
We also notice that for each subset $Y \subset \omega_\lambda$, the set $U_Y = \prod_{n<\omega} U_n$ where $U_0 = Y$ and $U_i = \omega_\lambda$ otherwise. Then U_Y is open. Thus, there are at least $2^{\aleph_\lambda}$ open sets.

Cardinality of $\mathscr{B}$

Corollary

$|\mathscr{B}| = 2^{\aleph_\lambda}$

Proof

It follows from the fact that every regular open set is also open, and the all the sets U_Y constructed in the previous proposition are clopen.

References

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