Definitions

Definition. *A tree, $(T, <)$, is a partially ordered set such that $\forall t \in T$, the set $\{t' \in T : t' < t\}$ is well ordered.*

A tree T is called (singularly) rooted if there exists $t \in T$ such that $t \leq t'$ for all $t' \in T$. This t is called the root of T.

For every $t \in T$ define the set

$$t^{<} = \{t' \in T : t' < t\}$$

Definition. *For every $t \in T$, let the height of the element t, denoted $\mathrm{ht}(t)$,to be the order type of the set $t^{<}$.*

Then, the height of the entire tree, $\mathrm{ht}(T)$, is defined to be $\sup(\mathrm{ht}(t) + 1 : t \in T)$. And, from this, we define another important set - the set of immediate successors of any element in our tree

$$t^{+} = \{t' \in T : \mathrm{ht}(t') = \mathrm{ht}(t) + 1 \text{ and } t < t'\}$$

Every tree T is a zero-dimensional topological space under the interval topology. This topology has base given by $(t', t] = \{s \in T : t' < s \leq t\}$ along with $\{t\}$ where t is the root of T. Under this topology, it is immediate that T is a Tychonoff space.

Definition. *If X is a Tychonoff space, then we say that a collection of real valued continuous functions on X, $\mathcal{G}$, is a generator for X if for every $x \in X$ and every closed set $K \subset X$ such that $x \notin K$, there exists $g \in \mathcal{G}$ such that $g(x) \notin \overline{g(K)}$.*

A generator $\mathcal{G}$ on a space X is said to be $(0,1)$ if for every $x \in X$ and every closed $K \subset X$ such that $x \notin K$, there exists $g \in \mathcal{G}$ such that $g(x) = 1$ and $g(K) = \{0\}$. Moreover, a $(0,1)$-generator $\mathcal{G}$ is called *super* if for every $g \in G$, the range of g is equal to $\{0, 1\}$.

The Result

Theorem. *If T is any rooted tree such that $|t^{+}| < \omega$ for all $t \in T$, then there exists a discrete* super *$(0,1)$-generator for T.*

Proof. Fix any rooted tree T such that $|t^{+}| < \omega$ for all $t \in T$.

Now, for all $t \in T$ such that $\mathrm{ht}(t)$ is 0 or a successor ordinal, let $g_t = \chi_{\{t\}}$. For any $t \in T$ such that $\mathrm{ht}(t)$ is a non-zero limit ordinal, we define $g_{t,\alpha} = \chi_{(t_\alpha, t]}$ where $\{t_\alpha : \alpha < \mathrm{ht}(t)\}$ is an order preserving enumeration of $t^{<}$. Now, let $\mathcal{G}$ be the collection of all the above defined functions.

The fact that $\mathcal{G}$ is a super $(0,1)$-generator is almost immediate. For any $t \in T$ and any closed $K \subset T$ such that $t \notin K$, then there exists a neighborhood $U_t = (t_\alpha, t]$, if t is of limit height, or $U_t = \{t\}$, if t is of successor height, such that $U_t \cap K = \emptyset$. Then, $g_{t,\alpha}$ or g_t will suffice to separate t from K.

The more careful property we need to prove is that $\mathcal{G}$ is a discrete subset of $\mathcal{C}(T, \mathbb{R})$ under the pointwise topology.

So, we break the next part into two steps.

Step 1: Pick any $g_t \in \mathcal{G}$. Allow $F = \{t\} \cup t^+$. Then define the neighborhood $U_{g_t} = B(g_t; F; \frac{1}{2})$.

Take any $g_{t'} \in \mathcal{G}$ where $t \neq t'$. Then $|g_t(t) - g_{t'}(t)| = 1 > \frac{1}{2}$. Hence, $g_{t'} \notin U_{g_t}$.

Now, take any $g_{t',\alpha} \in \mathcal{G}$. If $t \in (t'_\alpha, t']$, then find the successor $s \in t^+$ such that $s \in (t'_\alpha, t']$. Then we see that $|g_t(s) - g_{t',\alpha}(s)| = 1 > \frac{1}{2}$. If $t \notin (t'_\alpha, t']$, then $|g_t(t) - g_{t',\alpha}(t)| = 1 > \frac{1}{2}$. In either case, $g_{t',\alpha} \notin U_{g_t}$.

So, we have successfully shown that U_{g_t} witnesses discreteness as $U_{g_t} \cap \mathcal{G} = \{g_t\}$.

Step 2: Now, take any $g_{t,\alpha} \in \mathcal{G}$. We take $F = \{t, t_\alpha, t_{\alpha+1}\} \cup t^+$. We construct the neighborhood $U_{g_{t,\alpha}} = B(g_{t,\alpha}; F; \frac{1}{2})$. We now show that this nieghborhood witnesses discreteness.

Take any $g_{t'} \in \mathcal{G}$. Then, by definition, $|g_{t,\alpha}(t) - g_{t'}(t)| = 1 > \frac{1}{2}$. So, $g_{t'} \notin U_{g_{t,\alpha}}$.

Now, take any $g_{t',\beta} \in \mathcal{G}$. If $t = t'$, then either $|g_{t,\alpha}(t_\alpha) - g_{t',\beta}(t_\alpha)| = 1$ or $|g_{t,\alpha}(t_{\alpha+1}) - g_{t',\beta}(t_{\alpha+1})| = 1$. In either case, $g_{t',\beta}$ will not be in $U_{g_{t,\alpha}}$. Now, assume $t \neq t'$. If t and t' are incompatible or if $t' < t$, then $|g_{t,\alpha}(t) - g_{t',\beta}(t)| = 1 > \frac{1}{2}$. Now assume that $t' > t$. If $t \notin (t'_\beta t]$, then $|g_{t,\alpha}(t) - g_{t',\beta}(t)| = 1 > \frac{1}{2}$. And, in the other case, if $t \in (t'_\beta, t']$, then we find the successor $s \in t^+$ such that $s \in (t'_\beta, t']$. Then, we have $|g_{t,\alpha}(s) - g_{t',\beta}(s)| = 1 > \frac{1}{2}$.

Therefore, we have shown that this neighborhood, $U_{g_{t,\alpha}}$ witnesses discreteness as $U_{g_{t,\alpha}} \cap \mathcal{G} = \{g_{t,\alpha}\}$.

Thus, $\mathcal{G}$ is a discrete super $(0,1)$-generator for T. $\square$

The condition $|t + | < \omega$ for each $t \in T$ can be relaxed very slightly. Instead, we could have $|t^+| < \omega$ for all t such that there exists a $t' \in T$ where $t < t'$ with $\mathrm{ht}(t')$ a non zero limit ordinal. For example, the complete ω-ary tree of height ω has a discrete super $(0,1)$-generator while the complete ω-ary tree of height $\omega + 1$ does not.