# Functional countability of the square of the one-point compactification of the Kunen line minus the diagonal

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We will go over the construction of the Kunen line, functional countability of spaces, and start a proof to an answer posed in [1].

## 1 The Kunen Line

The Kunen Line first appeared in [2] by Juhász, Kunen, and Rudin. Their construction uses only $\mathsf{CH}$ to construct a hereditarily separable, non-Lindelöf, regular, first countable, locally compact, locally countable space of cardinality $\omega_1$. The method refines the Euclidean topology on $\mathbb{R}$.

### 1.1 Definitions

We state some definitions pertaining to the properties of the Kunen Line.

**Definition** (Local Cardinality)**.** *The local cardinality of a space $X$ is a cardinal $\kappa$ such that for every point $x \in X$, there exists an open neighborhood $U$ of $x$ such that $|U| \leq \kappa$.*

A space is said to be *Locally Countably* if the local cardinality of $X$ is $\omega$.

**Definition** (Hereditarily Separable (HS))**.** *A space $X$ is said to be hereditarily separable if every subspace $S \subset X$ has a countable dense subset.*

**Definition** (Continuum Hypothesis ($\mathsf{CH}$))**.** *The Continuum Hypothesis states that $\mathfrak{c} = \omega_1$.*

### 1.2 Construction

Let $\rho$ denote the usual topology on $\mathbb{R}$ and let $\tau$ denote the finer topology that will be constructed. Assume the $\mathsf{CH}$ and enumerate $\mathbb{R} = \{x_\alpha : \alpha < \omega_1\}$. Then consider all initial segments $X_\alpha = \{x_\beta : \beta < \alpha\}$. We let $\tau_\alpha$ and $\rho_\alpha$ be the topologies on the subspace $X_\alpha$. Also, enumerate the set $[\mathbb{R}]^\omega = \{S_\alpha : \alpha < \omega_1\}$.

We inductively build topologies $\tau_\eta$ for $\eta \leq \omega_1$ such that for all $\xi < \eta \leq \omega_1$ we have the following:

1. $\tau_\xi = \tau_\eta \cap \mathcal{P}(X_\xi$, i.e. $X_\xi$ is open in $\tau_\eta$
2. Each $\tau_\eta$ is first countable, locally compact, and $T_2$.

3. $\rho_\eta \subset \tau_\eta$

4. For each $\mu < \xi$, if $x_\xi \in \mathrm{Cl}_\rho(S_\mu)$ then $x_\xi \in \mathrm{Cl}_{\tau_\eta}(S_\mu)$

For each $\beta \leq \omega$, let $\tau_\beta = \mathcal{P}(X_\beta)$. Properties $1-4$ are rather trivial to check given that this is the discrete topology applied to countable sets.

Now, we assume we have constructed $\tau_\eta$ for all $\eta < \beta$ such that the above hold for all $\xi < \eta$.

If $\beta$ is a limit ordinal, we let $\tau_\beta = \{U \subset X_\beta : \forall \eta < \beta, (U \cap X_\eta \in \tau_\eta\}$. We check that the above four properties hold for all

# 2 Functional Countability

A space $X$ is functionally countable if $|f^{\rightarrow}(X)| \leq \omega$ for any real-valued continuous function $f : X \to \mathbb{R}$. An important proposition found in the paper is the following

**Proposition 2.1** (3.2)**.** *If $X$ is a space and $(X \times X) \setminus \Delta_X$ is functionally countable, then both spaces $X$ and $X \times X$ are functionally countable.*

It is also apparent that *scattered* spaces are are of massive importance to the property of functional countability. We give the definition of scattered and prove a few small

**Definition** (Scattered)**.** *A space $X$ is scattered if it does not contain a non-empty dense in itself subset.*

An immediate equivalent of is $X$ is scattered if every non-empty subset $S \subset X$ has an isolated point. So, $X$ is scattered if for all non-empty $S \subset X$, there exists $x \in S$ such that $\{x\}$ is open in $S$.

**Proposition.** *A scattered space is automatically hereditarily scattered, i.e. every subspace of a scattered space is scattered.*

*Proof.* Let $X$ be a scattered space and let $S \subset X$ be a non-empty subspace. Consider any subset $H \subset S$. As $H \subset X$, we have there exists $x \in H$ and $U$ open in $X$ such that $H \cap U = \{x\}$. Now, clearly $S \cap U$ is a non-empty open set in $S$. Then, it is immediate that $H \cap (S \cap U) = (H \cap U) \cap S = \{x\} \cap S = \{x\}$. Therefore, $S$ is scattered. □

**Proposition.** *If $X$ is second countable and scattered, then $|X| \leq \omega$.*

*Proof.* Let $X$ is a second countable scattered space. Assume $X$ is uncountable. We will construct a subset $S \subset X$ such that any point of $S$ will have a neighborhood base that includes another point of $S$.

Let $\mathscr{B}$ be the countable base of $X$. Consider the subfamily $\mathscr{B}' = \{B \in \mathscr{B} : B \text{ is countable}\}$. Allow $S = (\bigcup \mathscr{B}')^c$. We note that $S$ is nonempty. This is because $\bigcup \mathscr{B}'$ is at most countable as it is a countable union of countable sets. As $X$ is uncountable, this means $|S| > \omega$.

Pick any point $x \in S$. Find any basic open set $U$ containing $x$. We need to notice that $U \notin \mathscr{B}'$. This is because $x \in S = (\bigcup \mathscr{B}')^c$, then $U$ cannot be a proper subset of $\bigcup \mathscr{B}'$ Hence, $U$ cannot be a member of the family $\mathscr{B}'$. So, we conclude that $|U| > \omega$.

As $\bigcup \mathscr{B}'$ is countable, it must be the case that $U \cap S$ must be uncountable. More importantly, this means there exists $x' \in U$ with $x' \in S$. Therefore, any basic open set $U$ will never have the property that $U \cap S = \{x\}$. This means that $X$ must be countable.

□

I'm not quite sure how helpful these propositions are to understanding functional countability, but they nonetheless seem important to understanding scattered spaces.

Now, we turn our attention to functional countability of ordinals. We adopt the proof of theorem 3.13 to show that $\omega_1^2 \setminus \Delta$ is not functionally countable.

**Theorem 2.2** (3.13). *Let $X$ be a linearly ordered compact space whose complement of the diagonal is functionally countable, then $X$ is countable and metrizable.*

**Proposition.** *The space $\omega_1^2 \setminus \Delta$ is not functionally countable.*

*Proof.* Let $\alpha$ be any countable successor ordinal. Then, there exists $\beta$ such that $\alpha = \beta + 1$. Let $\alpha_- = \beta$ and $\alpha_+ = \alpha + 1$. We call these the neighbors of $\alpha$.

It is immediate that there exists an uncountable set $D$ of successor ordinals such that at least one neighbor is a successor oridnal. In fact, just take $D$ to be all successor ordinals. Then for all $\beta$, $\beta + 1$ is a neighbor and itself a successor ordinal.

The above property means that we may find $\omega_1$ many successor ordinals $\{\alpha_\tau : \tau < \omega_1\}$ such that $\{\alpha_\tau, (\alpha_\tau)_+\} \cap \{\alpha_\xi, (\alpha_\xi)_+\} = \emptyset$ whenever $\tau \neq \xi$.

Then, we take the set $Q = \{(\alpha_\tau, (\alpha_\tau)_+) : \tau < \omega_1\} \subset \omega_1^2 \setminus \Delta\}$. It is immediate that $Q$ is open and discrete as each of these coordinate of any element of $Q$ is a successor - hence isolated in the order topology.

We now show that $\mathrm{Cl}(Q) \setminus Q \subset \Delta$. Take any $(\alpha, \beta) \in \omega_1^2 \setminus \Delta$. Then there exist disjoint open intervals $U$ and $V$ such that $\alpha \in U$ and $\beta \in V$. Moreover, $W = U \times V$ is an open neighborhood of $(\alpha, \beta)$.

We can have two cases:

1. Case 1: $\alpha < \beta$. Then $\forall \tau \in U, \forall \xi \in V, \tau < \xi$. If $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in U \times V$, then if $(\alpha_1)_+ \in V$, then $\alpha_2 < (\alpha_1)_+$. So we can conclude that $\alpha_2 \leq \alpha_1$. A similar argument yields that $\alpha_1 \leq \alpha_2$. Hence, $\alpha_1 = \alpha_2$. We may repeat the argument for $\beta_1$ and $\beta_2$. So we conclude that $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$. Hence $(U \times V) \cap Q$ can only have at most one element.

2. Case 2: $\beta < \alpha$. Then we have $\forall \tau \in U, \forall \xi \in V$, we have $\tau > \xi$. So, if $(U \times V) \cap Q$ is non empty, then if we have any $(\gamma_\tau, (\gamma_\tau)_+) \in U \times V$, then $\gamma_\tau > (\gamma_\tau)_+$, which is impossible. So we conclude that $(U \times V) \cap Q = \emptyset$

Therefore, we have proved that every $(\alpha, \beta) \in \omega_1^2 \setminus \Delta$ has an open neighborhood $W$ such that $|W \cap Q| \leq 1$. Therefore $\mathrm{Cl}(Q) \setminus Q \subset \Delta$. Thus $Q$ is closed in $\omega_1^2 \setminus \Delta$.

So $Q$ is an uncountable clopen discrete subspace of $\omega_1^2 \setminus \Delta$, and by proposition 3.1($c$), $\omega_1^2 \setminus \Delta$ is not functionally countable.

□

## 3 Functional Countability for the space $(X+1)^2 \setminus \Delta$

We now examine the possibility of the space $(X+1)^2 \setminus \Delta$ of being functionally countable where $X$ is the Kunen line, and $X+1$ is the one-point compactification of $X$. The most important properties of the Kunen line that will be used is local countability - that every element $x \in X$ has a countable neighborhood, and hereditary separability - $X$ and every subspace contains a countable dense subset. So, let $\tau$ be the topology of the Kunen line.

Allow $\infty$ to be the point added in the one-point compactification of $X$. We will consider any point $(x, y) \in (X+1)^2 \setminus \Delta$ where $x \neq y$ and neither $x$ or $y$ are the point $\infty$. By local countability, there exists countable open neighborhoods $U_x$ and $U_y$. Moreover, each of these $U_x$ and $U_y$ must meet the countable dense subset of $X$. These give rise to a countable open neighborhood in $(X+1)^2 \setminus \Delta$.

Now, we take any non constant sequence $((x,y)_n)_{n<\omega}$ such that $(x,y)_n \to (x,y)$ in the standard Euclidean topology. We hope to conclude that $(x,y)_n$ rarely converges to $(x,y)$.

By the previous remarks, it is quite easy to see that $(x,y)_n$ will converge in the Kunen line topology to $(x,y)$ only if the tail of the two coordinates of the sequence are subsets of $U_x$ and $U_y$. Hence, uncountably many sequences will not ever be convergent to $(x,y)$ in $\tau$.

## References


[1] Tkachuk, V. V. (2021). A Corson compact space is countable if the complement of its diagonal is functionally countable. Studia Scientiarum Mathematicarum Hungarica, 58(3), 398–407. https://doi.org/10.1556/012.2021.58.3.1508

[2] Juhász, I., Kunen, K., & Rudin, M. E. (1976). Two more hereditarily separable Non-Lindelöf Spaces. Canadian Journal of Mathematics, 28(5), 998–1005. https://doi.org/10.4153/cjm-1976-098-8