Diamond Equivalents

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We state the three forms of diamond that we will show are equivalent.

1. $\diamondsuit$: there is a family $\{f_\alpha\}_{\alpha<\omega_1}$ of functions such that f_α maps α into α, and, if f maps ω_1 into ω_1, then $\{\alpha : f|_\alpha = f_\alpha\}$ is stationary.
2. $\diamondsuit_1$: there is a family $\{S_\alpha\}_{\alpha<\omega_1}$ of subsets of ω_1 such that $S_\alpha \subset \alpha$ and, if $S \subset \omega_1$, then $\{\alpha : S \cap \alpha = S_\alpha\}$ is stationary.
3. $\diamondsuit_2$: there is a family $\{M_\alpha\}_{\alpha<\omega_1}$ of subsets of $\omega_1 \times \omega_1$ such that $M_\alpha \subset \alpha \times \alpha$ and, if $M \subset \omega_1 \times \omega_1$, then $\{\alpha : M \cap (\alpha \times \alpha) = M_\alpha\}$ is stationary.
4. $\diamondsuit_{2b}$: there is a family of functions $\{f_\alpha\}_{\alpha<\omega_1}$ where $f_\alpha : \alpha \times \alpha \to \alpha$ and if given any $f : \omega_1 \times \omega_1 \to \omega$ the set $\{\alpha : f|_{(\alpha\times\alpha)} = f_\alpha\}$ is stationary.

We will now show that the above formulations are all equivalent.

Proof. We now prove the following implications.

i. $\diamondsuit \implies \diamondsuit_1$: Define $S_\alpha = f_\alpha(\alpha)$. We show that S_α satisfy $\diamondsuit_1$. Now, let $S \subset \omega_1$. Then, S can be thought of as the image of some function $f : \omega_1 \to \omega_1$. Then, by the assumption, the collection of α such that f agrees with f_α is stationary. That means that $S \cap \alpha = S_\alpha$ for a stationary set of α as desired as f and f_α agree on α which means their images restricted to α will be equal.

ii. $\diamondsuit_1 \implies \diamondsuit_2$: let f be the bijection from ω_1 to $\omega_1 \times \omega_1$ such that $f^{\to}(\alpha) = \alpha \times \alpha$ for all limit α. Given a $\diamondsuit$ sequence $\{S_\alpha\}_{\alpha<\omega_1}$, let $M_\alpha = f^{\to}(S_\alpha)$ for limit α and $M_\alpha = \emptyset$ otherwise. By the above property, we will have that $M_\alpha \subset \alpha \times \alpha$.

   Let M be any subset of $\omega_1 \times \omega_1$. Then let $S = f^{-1}(M) \subset \omega_1$. By assumption, $\{\alpha : S \cap \alpha = S_\alpha\}$ is stationary. Then $\{\alpha \in \Lambda_{\omega_1} : S \cap \alpha = S_\alpha\}$ is also stationary as Λ_{ω_1} is a club. Then $\{\alpha \in \Lambda_{\omega_1} : f^{\to}(S) \cap f^{\to}(\alpha) = f^{\to}(S_\alpha)\} = \{\alpha \in \Lambda_{\omega_1} : M \cap (\alpha \times \alpha) = M_\alpha\}$ is stationary. Hence $\{\alpha : M \cap (\alpha \times \alpha) = M_\alpha\}$ is stationary as it contains a stationary set.

iii. $\diamondsuit_2 \implies \diamondsuit$: Let $A = \{\alpha < \omega_1 : M_\alpha \text{ is a function}\}$. Then, let $f_\alpha = M_\alpha$ for all $\alpha \in A$ and let $f_\alpha = 0$ for all $\alpha \notin A$. Now, let $f : \omega_1 \to \omega_1$ be any function. Then $f = M \subset \omega_1 \times \omega_1$. By assumption $\{\alpha : M \cap (\alpha \times \alpha) = M_\alpha\}$ is a stationary set. We notice that $M \cap (\alpha \times \alpha)$ is simply the restriction of $M = f$ to domain α and codomain α. Then $f|_\alpha = f_\alpha$ on a stationary set.

iv. $\diamondsuit_{2b} \implies \diamondsuit$ define the family of functions $\{g_\alpha\}_{\alpha<\omega_1}$ by $g_\alpha = f_\alpha \circ i_\alpha$ where $i_\alpha : \alpha \to \alpha \times \alpha$ by $i_\alpha(\beta) = (\beta, \beta)$. Let $f : \omega_1 \to \omega_1$ be any function. Then, we can extend f to a function $\overline{f} : \omega_1 \times \omega_1 \to \omega_1$ by $\overline{f}((\alpha, \alpha)) = f(\alpha)$ for all $(\alpha, \alpha) \in \Delta_{\omega_1 \times \omega_1}$ and $\overline{f}((\beta, \gamma)) = 0$ for all $(\beta, \gamma) \notin \Delta_{\omega_1 \times \omega_1}$.

By assumption, the set $\{\alpha : \overline{f}|_{(\alpha\times\alpha)} = f_\alpha\}$ is stationary. So, this means $\{\alpha : f|_\alpha = g_\alpha\}$ is as well.

v. $\diamondsuit \implies \diamondsuit_{2b}$ Let ϕ be the bijection from ω_1 to $\omega_1 \times \omega_1$ that maps α onto $\alpha \times \alpha$ for all limit ordinals. Then, set $\phi_\alpha^{-1} = \phi^{-1}|_{\alpha\times\alpha} : \alpha \to \alpha \times \alpha$. Let $\{f_\alpha\}_{\alpha<\omega_1}$ be the sequence guaranteed in the assumption. Then construct the functions $\{g_\alpha\}_{\alpha<\omega_1}$ by $g_\alpha = f_\alpha \circ \phi_\alpha^{-1} : \alpha \times \alpha \to \alpha$ for limit α and $g_\alpha = 0$ for any other.

Let $f : \omega_1 \times \omega_1 \to \omega_1$ be any function. Then $f \circ \phi : \omega_1 \to \omega_1$. By assumption, $\{\alpha : f \circ \phi|_\alpha = f_\alpha\}$ is stationary. Restrict the above set to Λ_{ω_1}. Then $\{\alpha \in \Lambda_{\omega_1} : f|_{(\alpha\times\alpha)} = f_\alpha \circ \phi^{-1}|_\alpha\}$ will also be stationary.

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