Boolean algebras of arbitrary size

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We begin this section with a review of product spaces. We will define the Baire Space of Weight λ, and use that space in order to construct a complete Boolean Alegbra of arbitrarily large size that is also countably generated.

Definition 1 (Cartesian Product). Let A be any set, and for every $\alpha \in A$, let X_α be a nonempty set. We define the Cartesian Product as the following set

$$\prod_{\alpha \in A} X_\alpha = \left\{ f : A \to \bigcup_{\alpha \in A} X_\alpha \middle| (\forall \alpha \in A) f(\alpha) \in X_\alpha \right\}$$

In words, the Cartesian product is the collection of all functions from our indexing set such that the image at each index α lies in X_α. If all X_α are all the same set, we may instead write the product simply as X^A. For every product, $\prod_{\alpha \in A} X_\alpha$, there exists a mapping onto each factor called the projection. This map is denoted π_α and is defined by $f \mapsto f(\alpha)$.

In the case when all the sets X_α that we are considering are also topological spaces, we may hope equip the product of those spaces with a natural enough topology. We would, in particular, like a the smallest topology on $\prod_{\alpha \in A} X_\alpha$ such that each projection map, π_α, is continuous.

Definition 2 (Tychonoff Topology). Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces. Then consider the product $\prod_{\alpha \in A} X_\alpha$.We define the Tychonoff Topology on the product as as the topology which has as its subbase the collection

$$\mathscr{B}_S = \{\pi_\alpha^{-1}(U) : \alpha \in A \text{ and } U \text{ is open in } X_\alpha\}$$

The Tychonoff Topology (or from here on out, the product topology) then has as a base all finite intersections of subbasic elements. But what do these sets really look like? Well, to understand, we first look at subbasic elements. For any $\beta \in A$ and U open in X_β, the set $\pi_\beta^{-1}(U)$ will be just the collection of functions from the product space such that $f(\beta) \in U$. Hence, it will just be the product $\prod_{\alpha \in A} S_\alpha$ where $S_\beta = U$ and $S_\alpha = X_\alpha$ for all $\alpha \neq \beta$. Then taking finite intersections of subbasic sets gives us that sets of the form $\prod_{\alpha \in A} U_\alpha$ where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many α form a base for the product topology.

Now, we define the main space of interest which will be used to construct our desired Boolean Algebra.

Definition 3 (Baire Space of Weight-λ). For any ordinal λ, we define the Baire Space of Weight-λ to be the product space $\prod_{n<\omega} \omega_\lambda = (\omega_\lambda)^\omega = \omega_\lambda^\omega$ where each ω_λ is equipped with the discrete topology.

Throughout the rest of this section, let X be the Baire Space of Weight-λ. From our definitions, we see that X is really a set of functions f with domain ω and codomain ω_λ. And since ω_λ has the discrete topology, any subset of ω_λ is open. Then we see that the subbasic sets of ω_λ^λ take the general form $\prod_{n<\omega} U_n$ where $U_n = \omega_\lambda$ for all but one m and U_m is a nonempty subset of ω_λ.

These Baire Spaces are of interest in their own right, and many of the properties they possess are helpful in proving the existence of our arbitrarily large and countably generated complete Boolean Algebra. So we begin with an examination of X. But first we give the definition for a refinement of a cover.

Definition 4 (Refinement of a Cover). Let Y be a topological space with two open covers $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ and $\mathscr{V} = \{V_\beta\}_{\beta\in B}$. We say that $\mathscr{V}$ is a refinement of the cover $\mathscr{U}$ (in symbols, $\mathscr{V} \prec \mathscr{U}$) if for all $V_\beta \in \mathscr{V}$ there is some $U_\alpha \in \mathscr{U}$ such that $V_\beta \subset U_\alpha$.

Baire Spaces have an important property of refinements: it contains a countable sequence of disjoint clopen covers such that each successive cover is a refinement of the previous. In this context, a cover $\mathscr{U}$ of X is clopen and disjoint if all $U \in \mathscr{U}$ is clopen and any two different elements of the cover are disjoint, respectively. In pursuit of showing this, we define the sets $U(n, f) = \{g \in X : \forall m \leq n, f(m) = g(m)\}$ and $B_n = \{U(n, f) : f \in X\}$.

Proposition 1. *The set $\{B_n\}_{n<\omega}$ is a countable set of disjoint clopen covers of X such that for whenever $m < n < \omega$, $B_n \prec B_m$.*

Proof. We first show that for each $n < \omega$, B_n is a clopen cover. So, we see that each $U(n, f)$ is open as $U(n, f) = \prod_{i<\omega} U_i$ where $U_i = \{f(i)\}$ for $i \leq n$ and $U_i = \omega_\lambda$ for $i > n$. We also see that $U(n, f)^c = \prod_{i<\omega} U_i$ where $U_i = \omega_\lambda \setminus \{f(i)\}$ for $i \leq n$ and $U_i = \omega_\lambda$ for $i > n$ is open. And clearly B_n is a cover as for every $f \in X$, $f \in U(n, f)$. And, consider two functions such that $U(n, f) \neq U(n, g)$. Then clearly $f(m) \neq g(m)$ for some $m < n$. But this certainly means that there can never be a function h such that $h(m) = f(m)$ and $h(m) = g(m)$ for all $m < n$. Therefore $U(n, f) \bigcap U(n, g) = \emptyset$.

To see the rest, we only show $B_{n+1} \prec B_n$ and then use the transitivity of refinements to complete the proof. So, pick any $U(n+1, f) \in B_{n+1}$. Then certainly $U(n+1, f) \subset U(n, f) \in B_n$. As if any $g \in U(n + 1, f)$, then $g(m) = f(m)$ for all $m \leq n + 1$, then $g(m) = f(m)$ for all $m \leq n$. So we conclude that that B_{n+1} refineds B_n. □

Another important fact about these Baire Spaces is that they are indeed first-countable. Even more than that, the countable neighborhood base for every $f \in X$ is clopen.

Proposition 2. *The set $\{U(n, f) : n < \omega\}$ forms a clopen neighborhood base for every $f \in X$.*

Proof. Allow U to be a basic open set of X containing f. Then $U = \prod_{i<\omega} U_i$ where U_i is open in ω_λ for all i, and $U_i = \omega_\lambda$ for all but finitely many i. Allow N to be the smallest index such that $U_n = \omega_\lambda$ for all $n > N$. Then we see that $U(N, f) \subset U$. For let $g \in U(N, f)$. We have to show that $g(m) \in U_m$ for every $m < \omega$. If $m \leq N$, then $g(m) = f(m)$. Since $f \in U$, we see that $g(m) \in U_m$. Now if $m > N$, then $g(m) \in U_m$ as $U_m = \omega_\lambda$. Hence $g \in U$. □

Finally, the possibly most important property of Baire Spaces that we will need is the fact that the above sequence of disjoint clopen covers $\{B_n\}_{n<\omega}$ allows us to write every open set as a disjoint union of clopen sets. This a powerful fact that will enable us to handle the Boolean Algebra of the regular open sets of X in an easier manner.

Proposition 3. *Every open set $U \subset X$ can be written as a countable union of disjoint clopen sets.*

Proof. Define the function $\phi_U : U \to \omega$ as $\phi_U(f) = \inf(n < \omega : U(n, f) \subset U)$. This function is well defined defined by the previous proposition. Next, let $U_n = \bigcup\{U(n, f) : \phi_U(f) = n\}$. All U_n are pairwise disjoint. For any $f \in U$, if $f \in U_n$ and $f \in U_m$, then $n = \phi_U(f) = m$, which cannot happen unless $n = m$. It's also clear that each U_n is clopen as U_n is a union of basic open sets, and $X \setminus U_n$ is simply the union of the open elements of B_n that are not in U_n.

Finally, we see that

$$U = \bigcup_{n<\omega} U_n$$

For all $f \in U$, the set $\{n < \omega : U(n, f) \subset U\}$ is nonempty by proposition 1. Thus, $f \in U_n$ for some $n < \omega$, and $U \subset \bigcup_{n<\omega} U_n$. The reverse inclusion is clear as $U_n \subset U$ for all n. □

We are now ready to attack the main result of this section:

Theorem. *For every ordinal λ, there exists a countably generated Boolean Algebra of size 2^{ω_λ}.*

Allow $\mathscr{B}$ to be the Boolean Algebra of regular open sets of X. We give a reminder that a subset A of X to be regular open if and only if $A = (\bar{A})^\circ$, i.e. A is equal to the interior of its closure, and that the collection regular open sets, $\mathscr{B}$, of any topological space form a complete Boolean Algebra under the following Boolean Algebra operations: for any $\mathscr{U} \subset \mathscr{B}$,

$$\bigvee \mathscr{U} = \left(\overline{\bigcup \mathscr{U}}\right)^\circ$$

for any $\mathscr{U} \subset \mathscr{B}$ with $\mathscr{U} \neq \emptyset$

$$\bigwedge \mathscr{U} = \left(\bigcap \mathscr{U}\right)^\circ$$

and for any $U \in \mathscr{B}$,

$$U^c = (X \setminus U)^\circ$$

With the above relations in mind, we turn to show that $\mathscr{B}$ has a generally easy to write down set of generators. We define the set $A_{n,\eta} = \{f \in X : f(n) = \eta\}$ for every $n < \omega$ and $\eta < \omega_\lambda$.

Proposition 4. *The collection $\{A_{n,\eta} : n < \omega, \eta < \omega_\lambda\}$ generates $\mathscr{B}$.*

Proof. Let $U \in \mathscr{B}$ be a regular open set. Then U is also open. By proposition 3, we may write U as a countable union of disjoint clopen sets U_n. Let $\phi_U(f)$ be defined as in proposition 3. Then the proof follows from the following 3 facts.

i. For any $f \in X$,

$$U(n,f) = \bigwedge_{i\leq n} A_{i,f(i)}$$

ii. For every $n < \omega$,

$$U_n = \bigvee\{U(n,f) : \phi_U(f) = n\}$$

iii.

$$U = \bigvee_{n<\omega} U_n$$

From these three, we can conclude that any regular open set U may be written as meets and joins of the sets $A_{n,\eta}$.

To prove i, we see that $U(n,f) = \bigcap_{i\leq n} A_{i,f(i)}$. And since $U(n,f)$ is clopen, it is regular open. Hence

$$U(n,f) = (U(n,f))^\circ = \left(\bigcap_{i\leq n} A_{i,f(i)}\right)^\circ = \bigwedge_{i\leq n} A_{i,f(i)}$$

To prove ii, we see that $U_n = \bigcup\{U(n,f) : \phi_U(f) = n\}$. Again, U_n is clopen, so

$$U_n = \left(\overline{U_n}\right)^\circ = \left(\overline{\bigcup\{U(n,f) : \phi_U(f) = n\}}\right)^\circ = \bigvee\{U(n,f) : \phi_U(f) = n\}$$

To prove iii, we see that $U = \bigcup_{n<\omega} U_n$ by proposition 3. As U is regular open, we get

$$U = \left(\overline{U}\right)^\circ = \left(\overline{\bigcup_{i<\omega} U_n}\right)^\circ = \bigvee_{i<\omega} U_n$$

□

Now we introduce a proposed countable set of generators for $\mathscr{B}$. For every $n, m < \omega$, we define the set $B_{n,m} = \{f \in X : f(n) \leq f(m)\}$. We must first show that these sets are in $\mathscr{B}$ to begin with.

Proposition 5. *For every $n, m < \omega$, $B_{n,m} \in \mathscr{B}$.*

Proof. It suffices to show that $B_{n,m}$ is clopen. For $\alpha, \beta < \omega_\lambda$, define the set $S_{\beta,\alpha} = \prod_{i<\omega} S_i$ where $S_n = \{\beta\}$, $S_m = \{\alpha\}$, and $S_i = \omega_\lambda$ otherwise. We note that $S_{\beta,\alpha}$ is open in the product topology. Then we see that

$$B_{n,m} = \bigcup_{\alpha<\omega_\lambda} \bigcup_{\beta\leq\alpha} S_{\beta,\alpha}$$

Therefore $B_{n,m}$ is open as it is the union of open sets.

Then, consider the complement $B^c_{n,m} = \{f \in X : f(n) > f(m)\}$. Let $T_{\beta,\alpha} = \prod_{i<\omega} T_i$ where $T_n = \{\beta\}$, $T_m = \{\alpha\}$, and $T_i = \omega_\lambda$ otherwise. Then $T_{\beta,\alpha}$ is open in the product topology. Then we see that

$$B^c_{n,m} = \bigcup_{\alpha<\omega_\lambda} \bigcup_{\alpha<\beta<\omega_\lambda} T_{\beta,\alpha}$$

Hence $B^c_{n,m}$ is open. □

We finally show that $\mathscr{B}$ is generated by our countable set.

Proposition 6. *The Boolean Algebra $\mathscr{B}$ is generated by $\{B_{n,m} : m, n < \omega\}$.*

Proof. We allow $\mathscr{B}'$ to be smallest complete Boolean Algebra contained in $\mathscr{B}$ that contain $B_{n,m}$ for all $n, m < \omega$. We will be done if we show that $A_{n,\eta} \in \mathscr{B}'$ for all $n < \omega$ and $\eta < \omega_\lambda$. We will do this by inducting on ω_λ. But first we make a few observations.

We define two sets $C_{n,\eta}$ and $Z_{n,\eta}$ as follows

$$C_{n,\eta} = \{f \in X : f(n) \leq \eta\}$$

and

$$Z_{n,\eta} = \{f \in X : f(n) < \eta\}$$

We see that both of these sets are clopen and also regular open. Moreover, they have the property that $A_{n,\eta} = C_{n,\eta} \bigwedge Z^c_{n,\eta}$. So, if we establish that each $C_{n,\eta}$ and $Z_{n,\eta}$ are in $\mathscr{B}'$, we are done. We start by fixing $n < \omega$.

For the base case, we set $\eta = 0$. Then we immediately see that $Z_{n,0} = \{f \in X : f(n) < 0\} = \emptyset \in \mathscr{B}'$. Moreover, we claim that

$$C_{n,0} = \bigwedge_{m<\omega} B_{n,m} = \left(\bigcap_{m<\omega} B_{n,m}\right)^\circ$$

If that equality is proven, we certainly have $C_{n,0} \in \mathscr{B}'$.

So, assume $f \in C_{n,0}$. We show that f is an interior point of $\bigcap_{m<\omega} B_{n,m}$. Well, certainly $A_{n,0}$ is an open set that contains f, and for any $g \in A_{n,0}$, we automatically get that $g(n) = 0 \leq g(m)$ for all $m < \omega$. Hence $f \in \bigwedge_{m<\omega} B_{n,m}$.

Let $f \in \bigwedge_{m<\omega} B_{n,m}$. By way of contradiction, assume $f(n) > 0$. By assumption, we can find an $N > n$ such that $U(N, f) \subset \bigcap_{m<\omega} B_{n,m}$. Then, we note that the function h defined as

$$h(m) = \begin{cases} f(m) & m \leq N \\ 0 & m > N \end{cases}$$

belongs to $U(N, f)$. Hence, h also belongs to $B_{n,N+1}$. This means that $h(n) \leq h(N+1) = 0$. But this is a contradiction as we assumed $h(n) > 0$. So, we get that $f(n) = 0$, and $f \in C_{n,0}$. Hence, we get that $A_{n,0} \in \mathscr{B}'$.

For the induction step, assume we've shown $A_{n,\xi} \in \mathscr{B}'$ for all $\xi < \eta$. Then we claim that

$$Z_{n,\eta} = \bigvee_{\xi<\eta} A_{n,\xi}$$

To see this, let $f \in Z_{n,\eta}$, then $f(n) < \eta$. So $f \in \bigcup_{\xi<\eta} A_{n,\xi}$. And if $f \in \bigcup_{\xi<\eta} A_{n,\xi}$, then $f \in A_{n,\lambda}$ for some $\lambda < \eta$. Hence $f(n) = \lambda < \eta$, and $f \in Z_{n,\eta}$. Therefore $Z_{n,\eta} = \bigcup_{\xi<\eta} A_{n,\xi}$, and since $Z_{n,\eta}$ is clopen, it follows that $Z_{n,\eta} = \left(\overline{Z_{n,\eta}}\right)^\circ = \left(\overline{\bigcup_{\xi<\eta} A_{n,\xi}}\right)^\circ = \bigvee_{\xi<\eta} A_{n,\xi}$. By assumption, we have that $Z_{n,\eta} \in \mathscr{B}'$.

For $n, m < \omega$ and $\eta < \omega_\lambda$, we define a new set

$$W_{n,m,\eta} = Z_{m,\eta} \bigcup B_{n,m}$$

. In easier words, $f \in W_{n,m,\eta}$ if and only if $f(m) < \eta$ or $f(n) < f(m)$. We notice that $W_{n,m,\eta}$ is certainly contained in $\mathscr{B}'$ as both $Z_{m,\eta}$ and $B_{n,m}$ are. To show $C_{n,\eta} \in \mathscr{B}'$, we show the equality of

$$C_{n,\eta} = \bigwedge_{m<\omega} W_{n,m,\eta} = \left(\bigcap_{m<\omega} W_{n,m,\eta} \right)^{\circ}$$

In the forward direction, we assume $f \in C_{n,\eta}$, then $f(n) \leq \eta$. Then, for each $m < \omega$, either $f(m) < \eta$ or $f(m) \geq \eta \geq f(n)$. So f will certainly either be in $Z_{m,\eta}$ or $B_{n,m}$ for each m, and we get that $f \in \bigcap_{m<\omega} W_{n,m,\eta}$. And since $C_{n,\eta}$ is clopen, we get that $C_{n,\eta} \subset \left(\bigcap_{m<\omega} W_{n,m,\eta}\right)^{\circ}$.

To show the reverse direction, we proceed by contrapositive. Assume $f \notin C_{n,\eta}$. Consider the set $U(N, f)$ for every $N > n$. The function h defined as

$$\begin{cases} f(m) & m \leq N \\ \eta & m > N \end{cases}$$

will lie in each $U(N, f)$. But $h \notin W_{n,N,\eta}$. So f is not an interior point of $\bigcap_{m<\omega} W_{m,n,\eta}$, and will not be contained in the interior.

Therefore, we get that $A_{n,\eta}$ is contained in $\mathscr{B}'$ for every $n < \omega$ and $\eta < \omega_\lambda$. Since $A_{n,\eta}$ generate $\mathscr{B}$, we can conclude that $\mathscr{B} = \mathscr{B}'$.

□

The only remaining step is to determine the size of $\mathscr{B}$. We will use the previous properties of Baire Spaces in order to bound the cardinality of the set of open sets in X to ascertain the size of $\mathscr{B}$.

Proposition 7. *The size of the set of all open sets in X is 2^{ω_λ}.*

Proof. From proposition 3, it's clear that every open set $U \subset X$ is a countable union of sets U_n such that U_n is the union of elements from B_n, the disjoint clopen cover of X. There are, at most, $(2^{\omega_\lambda})^n = 2^{\omega_\lambda}$ different combinations of elements from B_n that we can use to construct each U_n. Thus, every open set may be written in at most $(2^{\omega_\lambda})^\omega = 2^{\omega_\lambda \cdot \omega} = 2^{\omega_\lambda}$ ways.

We also notice that for all nonempty proper subsets $Y \subset \omega_\lambda$, the set $U_Y = \prod_{n<\omega} U_n$ where $U_0 = Y$ and $U_n = \omega_\lambda$ for $n > 0$ is open. There are indeed 2^{ω_λ} amount of these open sets. So we get that the cardinality of the collection open sets of X is precisely 2^{ω_λ}. □

We conclude the following corollary by two immediate facts. Every regular open set is also open, and the set U_Y constructed in the previous proposition is clopen and hence regular open.

Corollary. $|\mathscr{B}| = 2^{\omega_\lambda}$