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DISCRETE GENERATORS ON PSEUDOTREES

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Dedicated to the memory of Peter J. Nyikos.

Abstract. We study properties of branching and bounded chains in pseudotrees that imply local tree-like behavior. We define valent antichains, discrete branching, immediate branching, and MUB-valence. These properties abstract the behavior of the set of immediate successors in trees. For immediately branching pseudotrees, we show that failure of MUB-valence is equivalent to the existence of a point t such that ω^* embeds into $t^>$. Consequently, in immediately branching, MUB-valent pseudotrees, the subset $t^\geq$ is a tree. We then endow pseudotrees with the upper-limit topology. Under MUB-valence, we prove that the properties of being Dedekind complete, zero-dimensional, Hausdorff, and Tychonoff are equivalent. Finally, we characterize immediately branching, MUB-valent, Hausdorff pseudotrees admitting a binary generator.

1. Introduction

Pseudotrees are a natural relaxation of trees. They are partially ordered sets in which each point has a totally ordered set of predecessors. Pseudotrees have appeared in the study of pseudo-tree algebras, or initial chain algebras, their associated Stone spaces, and their general topological properties [1, 2, 6, 7]. One can recover trees from pseudotrees by requiring the predecessor sets to be well-ordered. That condition is global.

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We determine that tree structure can be recovered from local properties describing branching and the behavior of upper bounds for chains in pseudotrees.

We abstract the local branching properties of trees by defining valent antichains above points, which captures the direction of branching. If a valent antichain exists at every point in a pseudotree, we call the pseudotree discretely branching. We call a discretely branching pseudotree immediately branching when the branch toward each element of a valent set has a minimal point. We define MUB-valence of a pseudotree to be the property that the minimal upper bounds of every bounded chain are valent.

The first result shows that, for an immediately branching pseudotree, failure of MUB-valence is equivalent to the existence of a point t such that ω^* embeds into $t^>$. Consequently, if an immediately branching pseudotree is MUB-valent, then every interval $(s, t]$ is well-ordered, and the subpseudotree $t^\geq$ is a tree. The defining tree condition, namely that the set of predecessors being well-ordered, is not assumed. Yet it emerges from local branching and chain properties.

The second part of the paper studies the upper-limit topology on pseudotrees. This topology differs from the approaches in [7]. Instead of attempting to recover the usual order topology on chains as subspaces, the upper-limit topology is chosen to keep the increasing limit structure found in ordinals and trees for the class of pseudotrees.

In [4], it was shown that every ordinal admits a discrete $(0, 1)$-generator. Since ordinals are well-ordered trees, it is natural to conjecture whether that result holds for pseudotrees. We call a discrete $(0, 1)$-generator a binary generator if every member is $\{0, 1\}$-valued. We prove that an immediately branching, MUB-valent, Hausdorff pseudotree admits a binary generator if and only if it admits a homogeneous pair of cofinal sequences and local clopen neighborhood bases satisfying a finite separation property.

2. Order Structure of Pseudotrees

This section introduces order-theoretic properties on pseudotrees. We describe the branching at points by antichains in the set $t^>$. We classify pseudotrees by whether the branches above a point have minimal elements. Then we investigate analogous branching properties for bounded chains.

2.1. Conventions and Standard Definitions.

Throughout, we work in ZFC. We use standard topological and set-theoretic terminology from

[3,5,9]. The symbol "$\subsetneq$" denotes proper inclusion, and "$\subset$" denotes inclusion, not necessarily proper.

If α is an ordinal, α^* denotes α equipped with the reverse order. For $A \subset \alpha$, we say that A is *cofinal* in α if for every $\beta < \alpha$, there exists $\gamma \in A$ with $\beta \leq \gamma$. The *cofinality* of an ordinal is the cardinal $\mathrm{cf}(\alpha) = \min\{|A| : A \text{ cofinal in } \alpha\}$.

Let $(X, \leq)$ be a poset. Elements $x, y \in X$ are said to be *comparable* if either $x \leq y$ or $y \leq x$, and x and y are said to be *incomparable* otherwise. For a fixed $x \in X$, let

$$x^< := \{y \in X : y < x\} \qquad x^\leq := \{y \in X : y \leq x\}$$
$$x^> := \{y \in X : y > x\} \qquad x^\geq := \{y \in X : y \geq x\}$$

An element $x \in X$ is *minimal* if $x^< = \emptyset$ and is *maximal* if $x^> = \emptyset$. For elements $x, y \in X$, we define the intervals

$$(x, y) := x^> \cap y^< \qquad [x, y) := x^\geq \cap y^<$$
$$(x, y] := x^> \cap y^\leq \qquad [x, y] := x^\geq \cap y^\leq$$

If $Y \subset X$, then Y is itself a poset with the order inherited from X. For non-empty $Y \subset X$, an *upper bound* (resp. *lower bound*) for Y is an element $x \in X$ such that $y \leq x$ (resp. $x \leq y$) for all $y \in Y$.

A subset $C \subset X$ is called a *chain* if C is totally ordered. In this paper, *bounded chain* means a chain that is bounded above. X is *Dedekind complete* if every non-empty chain that is bounded above has a least upper bound.

A subset $A \subset X$ is called an *antichain* if the elements of A are pairwise incomparable. An antichain $A \subset X$ is said to be *maximal* in X if there is no antichain $B \subset X$ such that $A \subsetneq B$.

A *tree* is a partially ordered set $(T, \leq)$ such that for all $t \in T$, $t^<$ is well-ordered. If a tree T has a unique least point, that point is called the *root* of T.

For every $t \in T$, the *height of* t is $h(t) := \mathrm{otp}(t^<)$. The *height of* T is defined to be $h(T) := \sup\{h(t) + 1 : t \in T\}$.

For trees, $t^+ = \{s \in T : t < s, h(s) = h(t) + 1\}$ is the set of *immediate successors of* t. If $h(t)$ is a non-zero successor ordinal, then t^- is the *immediate predecessor of* t and is defined to be the unique $s \in T$ such that $s < t$ and $h(s) + 1 = h(t)$.

Definition 2.1. A *pseudotree* is a partially ordered set $(T, \leq)$ such that $t^<$ is totally ordered for every $t \in T$.

2.2. **Valence and Branching.** We now introduce valent sets and use them to define when a pseudotree discretely branches.

Definition 2.2. Let t be a point in a pseudotree T. The *valence at t* is defined to be

$$\lambda_t := \min\{|F| : F \text{ is a maximal antichain in } t^>\}.$$

Remark 2.3. The cardinal λ_t is well-defined because maximal antichains exist by Zorn's lemma. Thus the minimum is taken over a non-empty set of cardinalities.

Proposition 2.4. *A point t in a pseudotree is maximal if and only if $\lambda_t = 0$. Moreover, for all non-maximal points t, $\lambda_t \geq 1$.*

Proof. If t is maximal, then $t^> = \emptyset$. Hence, the only maximal antichain in $t^>$ is the empty set, and $\lambda_t = 0$. If $\lambda_t = 0$, then the empty set is a maximal antichain in $t^>$. Hence $t^> = \emptyset$, and t is maximal.

If t is non-maximal, then $t^> \neq \emptyset$. Zorn's lemma guarantees the existence of at least one non-empty maximal antichain in $t^>$. Hence, $\emptyset$ is not a maximal antichain in $t^>$. Therefore $\lambda_t \geq 1$. □

Definition 2.5. Let X be a poset and let $F \subset X$ be an antichain. We say that F is *valent with respect to X* if each $x \in X$ is comparable with a unique $f \in F$.

Proposition 2.6. *If F is valent with respect to X, then F is a maximal antichain.*

Proof. If F were not a maximal antichain, then there exists $x \in X \setminus F$ such that $F \cup \{x\}$ is an antichain. Hence, x is comparable with no element of F. This contradicts the definition of valence. □

Definition 2.7. For a point t in a pseudotree and an antichain $F \subset t^>$, we say that F is *valent at t* if it is valent with respect to $t^>$.

Proposition 2.8. *Let t be a point in a tree T. The set t^+ is valent at t.*

Proof. We show that t^+ is an antichain. Take distinct $t_1, t_2 \in t^+$. If t_1, t_2 were comparable, then either $t_1 < t_2$ or $t_2 < t_1$. Then either $h(t_1) < h(t_2)$ or $h(t_2) < h(t_1)$. This contradicts $h(t_1) = h(t_2) = h(t) + 1$ because $t_1, t_2 \in t^+$.

If t is maximal, then $t^+ = \emptyset$ is vacuously valent with respect to $t^> = \emptyset$.

Assume t is non-maximal. Take any $s \in t^>$. If $h(s) > h(t) + 1$, then there exists $t' \in s^<$ such that $h(t') = h(t) + 1$. Since $s^<$ is well-ordered and $t \in s^<$, $t < t'$. Hence, $t' \in t^+$.

If $h(s) = h(t) + 1$, then $s \in t^+$. Hence, every point of $t^>$ is comparable with some point in t^+.

Suppose $r > t$ were comparable with $t_1, t_2 \in t^+$. As t^+ is an antichain, $r \notin t^+$. Thus $h(r) > h(t) + 1$. Hence, $t_1, t_2 \in r^<$. But $r^<$ is totally ordered. This would imply that t_1 and t_2 are comparable, contradicting the fact that t^+ is an antichain. □

Definition 2.9. A point t is said to be *discretely branching* if there exists $F \subset t^>$ that is valent at t. Otherwise, t is said to be *indiscretely branching.*

Remark 2.10. Observe that all maximal points t are discretely branching since $t^> = \emptyset$ is vacuously valent with respect to $t^>$.

Lemma 2.11. *Let t be a discretely branching point in a pseudotree T. If F is valent at t, then $|F| = \lambda_t$.*

Proof. If $\lambda_t = 0$, then t is maximal by Proposition 2.4, and the only set that is valent at t is the empty set.

Assume t is non-maximal. Let F be valent at t. Then $\lambda_t \leq |F|$ because F is a maximal antichain in $t^>$. Let G be an arbitrary maximal antichain in $t^>$. Define $f : G \to F$ by $f(s) = s'$, the unique member of F comparable with s.

Suppose f is not surjective. Then there exists $s' \in F$ such that no element of G is comparable with s'. This would imply that $G \cup \{s'\}$ is an antichain, contradicting the maximality of G. Hence f is surjective, and $|F| \leq |G|$. Thus, $\lambda_t \leq |F|$ and for every maximal antichain G, $|F| \leq |G|$. Hence $|F| \leq \lambda_t$, and $\lambda_t = |F|$. □

Definition 2.12. Let t be a non-maximal, discretely branching point in a pseudotree T, and let $\mathcal{F}_t := \{F \subset t^> : F \text{ is valent at } t\}$. For $F \in \mathcal{F}_t$ and $s \in F$, define the *branch from t toward s* to be

$$B_t(s) := \{t' \in t^> : t' \text{ is comparable with } s\}.$$

Proposition 2.13. *Let t be a non-maximal, discretely branching point in a pseudotree T. For every $F \in \mathcal{F}_t$, the family $\mathcal{B} := \{B_t(s) : s \in F\}$ partitions $t^>$.*

Proof. For any $t' \in t^>$, there exists a unique $s \in F$ such that t' is comparable with s since F is valent at t. Thus, $\mathcal{B}$ covers $t^>$.

Let $s_1, s_2 \in F$. Assume $B_t(s_1) \cap B_t(s_2) \neq \emptyset$. Then any $t' \in B_t(s_1) \cap B_t(s_2)$ is comparable with both s_1 and s_2. Since F is valent, $s_1 = s_2$ and $B_t(s_1) = B_t(s_2)$. Thus, for any distinct $s_1, s_2 \in F$, $B_t(s_1) \cap B_t(s_2) = \emptyset$, and $\mathcal{B}$ is a disjoint family. □

Proposition 2.14. *Let t be a non-maximal, discretely branching point in a pseudotree T. If $F_1, F_2 \in \mathcal{F}_t$, then the families*

$$\mathcal{B}_1 = \{B_t(s) : s \in F_1\} \quad \text{and} \quad \mathcal{B}_2 = \{B_t(s) : s \in F_2\}$$

are equal.

Proof. Fix $s_1 \in F_1$, and let $s_2 \in F_2$ be the unique point of F_2 that is comparable with s_1, which exists by valence. Take $r \in B_t(s_1)$, so r is comparable with s_1.

Suppose $r \leq s_1$. We claim that r is comparable with s_2. If $s_2 \leq s_1$, then $s_1^{\leq}$ is a chain, and hence either $s_2 \leq r$ or $r \leq s_2$. If $s_1 \leq s_2$, then transitivity guarantees that $r \leq s_2$.

Next assume that $s_1 \leq r$ and that r is not comparable with s_2. Let $r' \in F_2$ be the unique point of F_2 comparable with r. Since $s_1 \leq r$, the point s_1 is comparable with r' whenever $r \leq r'$. In that case, both s_2 and r' are elements of F_2 comparable with s_1. Thus $r' = s_2$, contradicting the assumption that r is not comparable with s_2.

If instead $r' \leq r$, then $r', s_1 \in r^{\leq}$, and hence r' and s_1 are comparable. Again, both s_2 and r' are members of F_2 comparable with s_1, so $r' = s_2$. Then $s_2 \leq r$, again contradicting the assumption that r is not comparable with s_2.

Therefore r is comparable with s_2, $r \in B_t(s_2)$, and $B_t(s_1) \subset B_t(s_2)$. The reverse inclusion is symmetric, and $B_t(s_1) = B_t(s_2)$.

Then every $B_t(s) \in \mathcal{B}_1$ is also an element of $\mathcal{B}_2$ and vice versa. Thus $\mathcal{B}_1 = \mathcal{B}_2$. □

Proposition 2.15. *Let t be a non-maximal, discretely branching point in a pseudotree T, and let F be any valent set at t. If $C \subset t^{>}$ is any non-empty chain, then there exists $s \in F$ such that $C \subset B_t(s)$.*

Proof. Fix a non-empty chain $C \subset t^{>}$. If C is a singleton, we are done. Otherwise, fix two distinct $c_1, c_2 \in C$, and suppose $c_1 \leq c_2$. Let $s_1, s_2 \in F$ be the unique members of F comparable with c_1 and c_2, respectively. Since c_2 is comparable with s_2, either $c_2 \leq s_2$ or $s_2 \leq c_2$.

If $c_2 \leq s_2$, then $c_1 \leq c_2 \leq s_2$, so c_1 is comparable with s_2. If $s_2 \leq c_2$, then $c_1, s_2 \in c_2^{\leq}$. Hence c_1 is comparable with s_2 since $c_2^{\leq}$ is a chain.

In either case, it follows that c_1 is comparable with both s_1 and s_2. Hence, the valence of F forces $s_1 = s_2$. Thus $c_1, c_2 \in B_t(s_1)$.

Then, fix any $c_0 \in C$. Let $s_0 \in F$ be the unique element comparable with c_0. Then, for any other $c' \in C$, apply the same argument for the pair $\{c_0, c'\}$. Therefore $c' \in B_t(s_0)$, and $C \subset B_t(s_0)$. □

Proposition 2.16. *Let t be a non-maximal, discretely branching point in a pseudotree T. Then there exists a partition*

$$\mathcal{R}_t = \{B_\lambda \subset t^{>} : \lambda < \lambda_t\}$$

of $t^{>}$ such that for every valent set F at t, there is a bijection $\tau : F \to \mathcal{R}_t$ such that $\tau(s) = B_t(s)$.

Proof. Choose F_1 to be any valent set at t. Hence, by Proposition 2.13, $\mathcal{B}_1 = \{B_t(s) : s \in F_1\}$ is a partition of $t^>$. By Lemma 2.11, $|F_1| = \lambda_t$. Let $i : \lambda_t \to F_1$ be a bijection. Set $B_\lambda = B_t(i(\lambda))$ for each $\lambda < \lambda_t$ and define $\mathcal{R}_t = \{B_\lambda : \lambda < \lambda_t\}$.

Now let F_2 be any valent set at t. By Proposition 2.14, the following families are equal

$$\{B_t(s) : s \in F_2\} = \{B_t(s) : s \in F_1\}.$$

For each $s \in F_2$, let $j(s)$ be the unique element of F_1 such that $B_t(s) = B_t(j(s))$. Set $\lambda_s = i^{-1}(j(s))$ and define $\tau(s) = B_{\lambda_s}$.

Then $\tau : F_2 \to \mathcal{R}_t$ is a bijection and

$$\tau(s) = B_{\lambda_s} = B_t(i(i^{-1}(j(s)))) = B_t(j(s)) = B_t(s)$$

for every $s \in F_2$ as desired. $\square$

Definition 2.17. Let t be a non-maximal, discretely branching point in a pseudotree T. The partition

$$\mathcal{R}_t = \{B_\lambda : \lambda < \lambda_t\}$$

is the set of *canonical branches above* t.

Remark 2.18. The word "canonical" is used to describe the independence of the partition from the choice of valent set. Indexing by λ_t, however, uses AC.

Definition 2.19. A pseudotree T is *discretely branching* if each point is discretely branching. T is *indiscretely branching* if at least one point is indiscretely branching.

Example 2.20. There is a pseudotree that is not discretely branching. Let

$$T = \{0\} \cup (\omega \times \{0, 1\})$$

ordered by the transitive closure of the relations

(i) $0 < (n, i)$ for every $(n, i) \in \omega \times \{0, 1\}$,
(ii) $(n, 0) < (m, 0)$ if and only if $n > m$, and
(iii) $(n, 0) < (n, 1)$.

Note that $(n, 0) < (m, 1)$ whenever $n \geq m$ after the transitive closure is taken, and $\{(n, 0) : n < \omega\}$ is totally ordered.

Since $(n, 0)^< = \{0\} \cup \{(m, 0) : m > n\}$ and $(n, 1)^< = \{0\} \cup \{(m, 0) : m \geq n\}$ are totally ordered, T is a pseudotree.

T does not discretely branch at 0. Suppose F is valent at 0. Since $\{(n, 0) : n < \omega\}$ is a chain, there can be at most one $(m, 0) \in F$.

If $F \cap \{(n, 0) : n < \omega\} = \emptyset$, then valence would force $F = \{(n, 1) : n < \omega\}$. However, $(2, 0)$ would be comparable with both $(1, 1)$ and $(0, 1)$.

If $F \cap \{(n,0) : n < \omega\} = \{(m,0)\}$, then valence forces $\{(n,1) : n > m\} \subset F$, and $(m+1,0)$ would be comparable with both $(m,0)$ and $(m+1,1)$.

In either case, we obtain a contradiction to the valence of F, and T does not discretely branch at 0.

2.3. Immediate Branching and Immediate Successors.

Valence detects the existence of branches above a point, but it does not require those branches to have least elements. Thus we define immediately branching pseudotrees.

Definition 2.21. Let t be a non-maximal, discretely branching point in a pseudotree T. We say that t is *immediately branching* if, for every $B_\lambda \in \mathcal{R}_t$, B_λ has a minimal element.

Proposition 2.22. *Let t be a non-maximal, discretely branching point in a pseudotree T. The point t is immediately branching if and only if for every valent set F at t, $B_t(s)$ has a minimal element for each $s \in F$.*

Proof. Let F be any valent set at t, and let $\tau : F \to \mathcal{R}_t$ be the bijection provided by Proposition 2.16.

Under this bijection, B_λ has a minimal element for each $\lambda < \lambda_t$ if and only if $B_t(s)$ has a minimal element for each $s \in F$. $\square$

Proposition 2.23. *Let t be a non-maximal, immediately branching point of a pseudotree T. Then every minimal element of $B_\lambda \in \mathcal{R}_t$ is the least element of B_λ.*

Proof. Let F be any valent set at t, let $B_\lambda \in \mathcal{R}_t$, and choose $s \in F$ such that $B_\lambda = B_t(s)$.

Let m_λ be a minimal element of B_λ. Since $s \in B_\lambda$, the points m_λ and s are comparable. By minimality, $m_\lambda \leq s$.

Take any $r \in B_\lambda$. Then r is comparable with s. If $r \leq s$, then both $r, m_\lambda \in s^{\leq}$. Hence r and m_λ are comparable as $s^{\leq}$ is totally ordered. By minimality, $m_\lambda \leq r$. If $s \leq r$, then transitivity gives $m_\lambda \leq r$.

Thus $m_\lambda \leq r$ for all $r \in B_\lambda$. Therefore m_λ is the least element of B_λ. $\square$

Definition 2.24. Let t be a point in a pseudotree T.

(i) The *set of immediate successors* of t is
$$t^+ := \{s \in t^> : \text{there is no } t' \text{ with } t < t' < s\}.$$

(ii) The point t is an *immediate successor* if there exists $s < t$ such that $t \in s^+$.

(iii) If t is an immediate successor, the *immediate predecessor*, denoted t^-, is the point s such that $s < t$ and $t \in s^+$.

Proposition 2.25. *If t is an immediate successor, then the immediate predecessor t^- is unique.*

Proof. Assume otherwise. Then there are $s_1, s_2 \in T$ such that $t \in s_1^+$ and $t \in s_2^+$. Since $t^<$ is totally ordered, s_1 is comparable with s_2. Say $s_1 \leq s_2$. If $s_1 < s_2$, then $s_1 < s_2 < t$. This contradicts that $t \in s_1^+$. Hence, $s_1 = s_2$, and the immediate predecessor is unique. □

Remark 2.26. If T is a tree, the above definitions coincide with the usual ones involving height.

Proposition 2.27. *For any point t in a pseudotree T, t^+ is an antichain.*

Proof. If $|t^+| \leq 1$, this is immediate. Assume there exist distinct $t_1, t_2 \in t^+$. If t_1 and t_2 were comparable, then either $t_1 < t_2$ or $t_2 < t_1$. Say $t_1 < t_2$. Then $t < t_1 < t_2$, which contradicts $t_2 \in t^+$. The case $t_2 < t_1$ is symmetric. Therefore t_1 is incomparable with t_2. □

Proposition 2.28. *Let t be a non-maximal, immediately branching point in a pseudotree T. For each $\lambda < \lambda_t$, let m_λ be the least element of B_λ. Then*

$$t^+ = \{m_\lambda : \lambda < \lambda_t\}$$

and t^+ is valent at t.

Proof. For each $\lambda < \lambda_t$, let m_λ be the least element of B_λ. Set

$$S = \{m_\lambda : \lambda < \lambda_t\}.$$

To show $S \subset t^+$, assume there exists t' with $t < t' < m_\lambda$. Then $t' \in B_\xi$ for some $\xi < \lambda_t$, and hence $m_\xi \leq t' < m_\lambda$.

If $\xi = \lambda$, then both $m_\lambda \leq t'$ and $t' < m_\lambda$, a contradiction. If $\xi \neq \lambda$, then the chain $C = \{m_\xi, t', m_\lambda\}$ has non-empty intersection with both canonical branches B_λ and B_ξ, contradicting Proposition 2.15. Thus no such t' exists, and $m_\lambda \in t^+$. Hence $S \subset t^+$.

To show $t^+ \subset S$, let $t' \in t^+$. Then $t' \in B_\lambda$ for some $\lambda < \lambda_t$. Hence $m_\lambda \leq t'$ by minimality. Since $t < t'$ and t' is an immediate successor of t, we must have $m_\lambda = t'$. Hence $t^+ \subset S$.

Finally, let $r \in t^>$. Then there is $\lambda < \lambda_t$ with $r \in B_\lambda$. Thus $m_\lambda \leq r$ by minimality, so r is comparable with $m_\lambda \in t^+$. If r is also comparable with $m_\xi \in t^+$, then $r < m_\xi$ is impossible because $m_\xi \in t^+$. Hence $m_\xi \leq r$. Thus $m_\lambda, m_\xi \in r^\leq$. Since $r^\leq$ is a chain, they are comparable. However, t^+ is an antichain by Proposition 2.27. Hence $m_\lambda = m_\xi$ and r is comparable with a unique $m_\lambda \in t^+$. Therefore t^+ is valent at t. □

Definition 2.29. We say that a discretely branching pseudotree T is *immediately branching* if every non-maximal point $t \in T$ is immediately branching.

Example 2.30. There is a discretely branching pseudotree that is not immediately branching. Let $T = \mathbb{R}$ with the usual order. T is discretely branching. For each $x \in T$, $\{x + 1\}$ is valent at x. However, for $\lambda = 0$, $B_\lambda = (x, \infty)$ has no minimal element.

Proposition 2.31. *Let T be a tree. Then T is immediately branching.*

Proof. Let t be a non-maximal point in T, and let $\mathcal{R}_t = \{B_\lambda : \lambda < \lambda_t\}$ be the canonical branches above t.

From Proposition 2.8, t^+ is valent. Hence, from Proposition 2.16, there is a bijection $\tau : t^+ \to \mathcal{R}_t$ such that for each $t' \in t^+$, $\tau(t') = B_t(t')$.

For any $\lambda < \lambda_t$, there is $t' \in t^+$ such that $\tau(t') = B_\lambda$. Now take any $b \in B_\lambda$. Since $b \in B_\lambda = \tau(t') = B_t(t')$, b is comparable with t'. If $b < t'$, then $t < b < t'$, contradicting $t' \in t^+$. Hence $t' \leq b$. Thus t' is minimal in B_λ. Therefore, for every $\lambda < \lambda_t$, B_λ has a minimal element, and T is immediately branching. $\square$

2.4. **MUB-Valence.** The preceding definitions describe branching above a single point. We now impose a similar condition for bounded chains, which we call MUB-valence.

Definition 2.32. Let T be a pseudotree, and let C be a non-empty bounded chain. We define the sets

$$\mathrm{UB}(C) := \{t \in T : \forall c \in C, c \leq t\}$$

and

$$\mathrm{MUB}(C) := \{m \in \mathrm{UB}(C) : m \text{ is minimal in } \mathrm{UB}(C)\}.$$

These are, respectively, the set of upper bounds of C and the set of minimal upper bounds of C.

Proposition 2.33. *If C is a bounded chain, then* $\mathrm{MUB}(C)$ *is an antichain.*

Proof. If $\mathrm{MUB}(C)$ had two distinct comparable points, m_1 and m_2, then either $m_1 < m_2$ or $m_2 < m_1$. In either case, this would contradict the minimality of either m_1 or m_2 in $\mathrm{UB}(C)$. $\square$

Definition 2.34. We say that T is *MUB-valent* if for every non-empty bounded chain $C \subset T$, $\mathrm{MUB}(C)$ is valent with respect to $\mathrm{UB}(C)$.

Proposition 2.35. *If T is Dedekind complete, then T is MUB-valent. The converse is false.*

Proof. Let T be Dedekind complete, and let C be any non-empty bounded chain in T. Then C has a least upper bound $m \in T$. The set $\{m\}$ is valent with respect to $\mathrm{UB}(C)$, since every element of $\mathrm{UB}(C)$ is comparable with

m. Moreover, $\mathrm{MUB}(C) = \{m\}$, because every upper bound for C lies above m. Thus T is MUB-valent.

Consider the tree $T = \omega \cup \{\infty_1, \infty_2\}$, ordered by the usual order on ω together with $n < \infty_i$ for all $n < \omega$ and $i = 1, 2$, and with ∞_1 and ∞_2 incomparable.

If $C \subset \omega$ is infinite, then the upper bounds of C are $\{\infty_1, \infty_2\}$, and these are minimal upper bounds and valent with respect to $\mathrm{UB}(C)$.

If $C \subset \omega$ is finite and non-empty, then C has a maximum in ω. If C contains one of ∞_1 or ∞_2, then C contains its least upper bound. Hence, this tree is MUB-valent, but it is not Dedekind complete since the bounded chain $C = \omega$ has no least upper bound. □

Proposition 2.36. *Every tree is MUB-valent.*

Proof. Fix a tree T and let $C \subset T$ be a non-empty bounded chain. Take any $u \in \mathrm{UB}(C)$, and consider the set $W_u := \mathrm{UB}(C) \cap u^{\leq}$. Since T is a tree, W_u is well-ordered because $u^{\leq}$ is well-ordered. Thus it has a least element, w_u. Moreover, $w_u \in \mathrm{MUB}(C)$. Indeed, if $v \in \mathrm{UB}(C)$ and $v < w_u$, then $v \in W_u$, contradicting the choice of w_u. Thus, for any $u \in \mathrm{UB}(C)$, we may find $w_u \in \mathrm{MUB}(C)$ such that $w_u \leq u$.

Now assume that there were distinct $w_1, w_2 \in \mathrm{MUB}(C)$ such that u is comparable with both. Minimality in $\mathrm{UB}(C)$ gives $w_1 \leq u$ and $w_2 \leq u$. But $u^{\leq}$ is well-ordered. Thus either $w_1 \leq w_2$ or $w_2 \leq w_1$. This contradicts the fact that $\mathrm{MUB}(C)$ is an antichain. Hence $w_1 = w_2$, and every $u \in \mathrm{UB}(C)$ is comparable with exactly one $w_u \in \mathrm{MUB}(C)$. Thus $\mathrm{MUB}(C)$ is valent with respect to $\mathrm{UB}(C)$. □

2.5. Embeddings of ω^*.

Definition 2.37. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. We say that P *embeds* into Q if there exists an injection $f : P \to Q$ such that $x \leq_P y$ if and only if $f(x) \leq_Q f(y)$.

Lemma 2.38. *Let P be a totally ordered set that is not well-ordered. Then ω^* embeds into P.*

Proof. If P is not well-ordered, then there exists a subset $A \subset P$ with no least element. Choose $a_0 \in A$. Assume a_n has been chosen. Since A has no least element, choose $a_{n+1} \in A$ with $a_{n+1} < a_n$. The map $n \mapsto a_n$ is an order embedding of ω^* into P. □

Theorem 2.39. *Let T be an immediately branching pseudotree. Then T is not MUB-valent if and only if there exists a point $t \in T$ such that ω^* embeds into $t^{>}$.*

For the following lemmas, assume that T is an immediately branching pseudotree.

Lemma 2.40. *If T is not MUB-valent, then there exists a non-empty bounded chain C such that either*

(i) $\mathrm{MUB}(C)$ *is empty, or*
(ii) *there exists an upper bound for C that is not comparable with any minimal upper bound.*

Proof. Let C be a non-empty bounded chain such that $\mathrm{MUB}(C)$ is not valent with respect to $\mathrm{UB}(C)$. If $\mathrm{MUB}(C) = \emptyset$, then case (i) holds.

Assume that $\mathrm{MUB}(C)$ is non-empty. Then there is some $u \in \mathrm{UB}(C)$ for which there is not a unique element of $\mathrm{MUB}(C)$ comparable with u. Thus either u is comparable with no element of $\mathrm{MUB}(C)$, or there are distinct $s_1, s_2 \in \mathrm{MUB}(C)$ such that u is comparable with both.

In the latter case, minimality of s_1 and s_2 in $\mathrm{UB}(C)$ implies $s_1 \leq u$ and $s_2 \leq u$. Since $u^{\leq}$ is a chain, s_1 and s_2 are comparable, contradicting that $\mathrm{MUB}(C)$ is an antichain. Hence case (ii) holds. □

Lemma 2.41. *If T is not MUB-valent, then there exists a point $t \in T$ such that ω^* embeds into $t^>$.*

Proof. Let C be the non-empty bounded chain obtained from Lemma 2.40.

First assume that $\mathrm{MUB}(C) = \emptyset$. Then $\mathrm{UB}(C)$ has no minimal elements. Choose $u_0 \in \mathrm{UB}(C)$. If $u_n \in \mathrm{UB}(C)$ has been chosen, then choose $u_{n+1} \in \mathrm{UB}(C)$ with $u_{n+1} < u_n$. This is possible since u_n is non-minimal in $\mathrm{UB}(C)$. This gives a strictly decreasing sequence $\langle u_n : n < \omega \rangle$ in $\mathrm{UB}(C)$, and the map $n \mapsto u_n$ is an order embedding of ω^* into $\mathrm{UB}(C)$.

For any fixed $c \in C$, $c < u_n$ for every $n < \omega$. Indeed, each u_n is an upper bound for C, and equality $c = u_n$ would make c a maximum of C, hence a least upper bound and an element of $\mathrm{MUB}(C)$, contrary to $\mathrm{MUB}(C) = \emptyset$. Thus $\{u_n : n < \omega\} \subset c^>$ for every $c \in C$.

Now assume that $\mathrm{MUB}(C)$ is non-empty, and let $u \in \mathrm{UB}(C)$ be incomparable with every element of $\mathrm{MUB}(C)$. Since $u \notin \mathrm{MUB}(C)$, u cannot be minimal in $\mathrm{UB}(C)$. Hence $u^< \cap \mathrm{UB}(C) \neq \emptyset$. Choose any $u_0 \in u^< \cap \mathrm{UB}(C)$. If $u_n \in \mathrm{UB}(C)$ with $u_n < u$ is given, choose $u_{n+1} \in u_n^< \cap \mathrm{UB}(C)$. Such a choice is possible because otherwise u_n would be minimal in $\mathrm{UB}(C)$, hence an element of $\mathrm{MUB}(C)$ comparable with u.

This produces a strictly decreasing sequence $\langle u_n : n < \omega \rangle \subset \mathrm{UB}(C)$ with $u_n < u$ for all n. For any $c \in C$, equality $c = u_n$ would imply that u_n is a maximum of C, hence an element of $\mathrm{MUB}(C)$ comparable with u, a contradiction. Thus $c < u_n$ for every $c \in C$ and every $n < \omega$.

Since $C \neq \emptyset$, fix $c \in C$. Then the sequence $\langle u_n : n < \omega \rangle$ lies above c, so ω^* embeds into $c^>$. □

Lemma 2.42. *If there exists a point $t \in T$ such that ω^* embeds into $t^>$, then T is not MUB-valent.*

Proof. Assume that ω^* embeds into $t^>$. Let $\langle u_n : n < \omega \rangle \subset t^>$ be the image of the embedding.

Since T immediately branches at t, Proposition 2.28 implies that t^+ is valent at t. For each n, choose $s_n \in t^+$ such that u_n and s_n are comparable. If $u_n < s_n$, then $t < u_n < s_n$, contradicting $s_n \in t^+$. Hence, $s_n \leq u_n$.

In fact $s_n < u_n$, since equality would give $t < u_{n+1} < u_n = s_n$, contradicting $s_n \in t^+$. Since all the points s_n lie in the chain $u_0^{\leq}$, they are comparable. Moreover, since t^+ is an antichain, all s_n are equal to some $s \in t^+$. Therefore $s < u_n$ for all $n < \omega$.

Define

$$C = \{t' \in B_t(s) : t' < u_n \text{ for all } n < \omega\}.$$

Then C is a non-empty bounded chain. It contains s, each u_n is an upper bound for C, and any two points of C lie in the chain $u_0^{\leq}$.

We show that $\mathrm{MUB}(C)$ is not valent with respect to $\mathrm{UB}(C)$. Suppose otherwise. Since each $u_n \in \mathrm{UB}(C)$, choose the unique $m_n \in \mathrm{MUB}(C)$ comparable with u_n. Minimality forces $m_n \leq u_n$, and in fact $m_n < u_n$, because $m_n = u_n$ would make u_{n+1} an upper bound for C strictly below m_n.

The points m_n all lie in the chain $u_0^{\leq}$. Since $\mathrm{MUB}(C)$ is an antichain, they are all equal to a single point $m \in \mathrm{MUB}(C)$.

Since $s, m \in u_0^{\leq}$, they are comparable. Since $s \in C$ and $m \in \mathrm{UB}(C)$, $s \leq m$. Hence $m \in B_t(s)$. Together with $m < u_n$ for every n, this gives $m \in C$. Thus m is the maximum of C.

Since m is not maximal in T as $m < u_0$, the set m^+ is non-empty and valent at m by Proposition 2.28. For each n, choose $m'_n \in m^+$ with m'_n comparable with u_n.

If $u_n < m'_n$, then $m < u_n < m'_n$, contradicting $m'_n \in m^+$. Therefore $m'_n \leq u_n$. Moreover, the inequality is strict. If $m'_n = u_n$, then $m < u_{n+1} < u_n = m'_n$ would again contradict $m'_n \in m^+$.

The points m'_n all lie in the chain $u_0^{\leq}$, and m^+ is an antichain, so there is $m' \in m^+$ such that $m'_n = m'$ for every $n < \omega$.

Since m is the maximum of C, every $c \in C$ satisfies $c \leq m < m'$, so $m' \in \mathrm{UB}(C)$. However, $m' < u_n$ for every n, and m' is comparable with s because both lie in $u_0^{\leq}$. Thus $m' \in B_t(s)$, and hence $m' \in C$. This contradicts that m is the maximum of C, since $m < m'$.

Therefore $\mathrm{MUB}(C)$ is not valent with respect to $\mathrm{UB}(C)$, and T is not MUB-valent. □

Proof of Theorem 2.39. The result follows from Lemmas 2.41 and 2.42. □

2.6. **Recovering Trees from Pseudotrees.** We prove that immediately branching, MUB-valent pseudotrees are locally tree-like.

Theorem 2.43. *Let T be an immediately branching, MUB-valent pseudotree. Then, for all $s, t \in T$ with $s < t$, the set $(s, t]$ is well-ordered. Consequently, $t^{\geq}$ is a tree.*

Proof. If the totally ordered set $(s, t] \subset s^{>}$ were not well-ordered, then Lemma 2.38 would give an order embedding of ω^* into $s^{>}$, contradicting Theorem 2.39.

Consider $t^{\geq}$ as a subposet. For every $t' \in t^{>}$, the set $(t, t']$ is well-ordered. The predecessor set of t' in the subposet $t^{\geq}$ is $[t, t')$ and is obtained from $(t, t']$ by adjoining t as a least element and deleting the maximum t'. Hence, $[t, t')$ is well-ordered.

If $t' = t$, then $(t, t'] = [t, t') = \emptyset$, which is also well-ordered. Hence $t'^{<} \cap t^{\geq}$ is well-ordered for every $t' \in t^{\geq}$. Therefore $t^{\geq}$ is a tree. $\square$

Corollary 2.44. *Let T be a pseudotree. Then T is immediately branching, MUB-valent, and has a unique least element if and only if T is a rooted tree.*

Proof. Let r be the unique least element of T. Then $T = r^{\geq}$, so Theorem 2.43 implies that T is a tree.

Conversely, if T is a tree with root r, then r is the unique least element and T is a pseudotree. Every tree is immediately branching by Proposition 2.31, and T is also MUB-valent by Proposition 2.36. $\square$

The preceding corollary summarizes our order-theoretic goals. The definition of a tree requires well-ordered predecessor sets, which is a global condition. For pseudotrees with a least element, immediate branching and MUB-valence recover this condition from only local branching and bounded chain properties.

3. The Upper-Limit Topology on Pseudotrees

We now endow pseudotrees with the upper-limit topology. The upper-limit topology is the topology generated by half-open intervals. On trees, this topology agrees with the usual interval topology as in [8]. On general pseudotrees, however, this topology behaves differently from the usual order topology.

3.1. Definition and Elementary Properties.

Definition 3.1. Let X be a partially ordered set. Define

$$\mathcal{S} = \{x^{>} : x \in X\} \cup \{x^{\leq} : x \in X\}.$$

The topology generated by $\mathcal{S}$ as a subbase is called the *upper-limit topology.*

Proposition 3.2. *Let T be a pseudotree endowed with the upper-limit topology. If t is non-minimal, then $\{(s,t] : s < t\}$ is a local base at t. If t is minimal, then t is isolated.*

Proof. Fix finite collections $\{s_i\}_{i=1}^n$ and $\{r_j\}_{j=1}^m$. Consider the basic open set

$$B = \left(\bigcap_{i=1}^n s_i^>\right) \cap \left(\bigcap_{j=1}^m r_j^\leq\right).$$

Suppose that B is a neighborhood of a non-minimal point $t \in T$.

If $n > 0$, then $s_i < t$ for each i, and the finite set $\{s_i : 1 \leq i \leq n\}$ has a maximum in the chain $t^\leq$. Let $s = \max\{s_i\}$. If $n = 0$, choose some $s < t$, which exists because t is non-minimal. If $m > 0$, then $t \leq r_j$ for each j, and hence $t^\leq \subset r_j^\leq$ for each j. Therefore, in all cases, $t \in (s,t] \subset B$.

If t is minimal, then $t^\leq = \{t\}$ is subbasic open, so t is isolated. □

Proposition 3.3. *A pseudotree T endowed with the upper-limit topology is a T_0 space.*

Proof. Fix two distinct points $s, t \in T$. Either $s^\leq$ or $t^\leq$ separates the points. If s and t are incomparable, either $s^\leq$ or $t^\leq$ works. If $s \leq t$, then $s^\leq$ works. If $t \leq s$, then $t^\leq$ suffices. □

Remark 3.4. For the remainder of the paper, every pseudotree is endowed with the upper-limit topology.

3.2. **Dedekind Completeness and Separation Axioms.** We adopt the convention that a space is *zero-dimensional* if it has a base of clopen sets, and that a space is *Tychonoff* if it is Hausdorff and completely regular. We show that Dedekind completeness implies the existence of a base of clopen sets and the Hausdorff and Tychonoff separation axioms. Conversely, under MUB-valence, all four properties are equivalent.

Theorem 3.5. *Let T be a pseudotree. If T is Dedekind complete, then T is also*

(i) *Hausdorff,*
(ii) *zero-dimensional,*
(iii) *Tychonoff.*

Proof. Let T be a Dedekind complete pseudotree.

First, we prove that T is Hausdorff. Let $t_1, t_2 \in T$ be distinct. If t_1 and t_2 are comparable, say $t_1 < t_2$, then $t_1^\leq$ and $(t_1, t_2]$ are disjoint open neighborhoods of t_1 and t_2, respectively.

Now assume that t_1 and t_2 are incomparable. If $t_1^{\leq}$ and $t_2^{\leq}$ are disjoint, then they separate t_1 and t_2. Otherwise, let $C = t_1^{\leq} \cap t_2^{\leq}$. Then C is a non-empty bounded chain, and Dedekind completeness gives $c = \sup C$. The neighborhoods $(c, t_1]$ and $(c, t_2]$ are disjoint. Indeed, any point in their intersection would be an element of C strictly above c.

Next, we show that T has a clopen base. If t is minimal, then $\{t\} = t^{\leq}$ is open and, since T is Hausdorff, closed. Assume t is non-minimal, and let $s < t$. It suffices to show that the subbasic open sets $s^{>}$ and $t^{\leq}$ are closed. Then each basic interval $(s, t] = s^{>} \cap t^{\leq}$ is clopen.

Let $r \in (t^{\leq})^c$. If $r > t$, then $(t, r]$ is an open neighborhood of r disjoint from $t^{\leq}$. If r and t are incomparable, consider $C = t^{\leq} \cap r^{\leq}$. If $C = \emptyset$, then $r^{\leq} \subset (t^{\leq})^c$ is open and misses t. If $C \neq \emptyset$, let $c = \sup C$. Then $c \leq t$ and $c \leq r$ since t and r are upper bounds of C. Since r and t are incomparable, $c < r$. Then $(c, r]$ is an open neighborhood of r missing $t^{\leq}$, because any point in $(c, r] \cap t^{\leq}$ would belong to C and be strictly above c. Hence $t^{\leq}$ is closed.

Let $r \in (s^{>})^c$. If $r \leq s$, then $s^{\leq}$ is a neighborhood of r missing $s^{>}$. If r and s are incomparable, consider $C = r^{\leq} \cap s^{\leq}$. If $C = \emptyset$, then $r^{\leq}$ is an open neighborhood of r contained in $(s^{>})^c$. If $C \neq \emptyset$, let $c = \sup C$. Then $c \leq s$ and $c \leq r$ since s and r are upper bounds of C. If $c = r$, then $r \leq s$, contradicting the assumption that r and s are incomparable. Thus $c < r$, and $(c, r]$ is an open neighborhood of r disjoint from $s^{>}$. Indeed, if $x \in (c, r] \cap s^{>}$, then $s < x \leq r$. By transitivity, $s < r$, contradicting that s and r are incomparable. Hence $s^{>}$ is closed.

Thus intervals of the form $(s, t]$ are clopen, and T has a base of clopen sets. Since every Hausdorff space with a clopen base is completely regular, T is Tychonoff. □

Lemma 3.6. *Let T be a Hausdorff MUB-valent pseudotree. Then T is Dedekind complete.*

Proof. Let C be a non-empty bounded chain. Then $\mathrm{UB}(C) \neq \emptyset$. Moreover, as the empty set is not valent with respect to $\mathrm{UB}(C)$, MUB-valence implies $\mathrm{MUB}(C) \neq \emptyset$.

If some $m \in \mathrm{MUB}(C)$ is minimal in T, then $C = \{m\}$, so m is the least upper bound of C, and we are done.

Now suppose that the minimal upper bounds of C are non-minimal in T, and $t_1, t_2 \in \mathrm{MUB}(C)$ are distinct. Then there are disjoint open neighborhoods U and V with $t_1 \in U$ and $t_2 \in V$ since T is Hausdorff. Choose $s_1 < t_1$ and $s_2 < t_2$ such that $(s_1, t_1] \subset U$ and $(s_2, t_2] \subset V$.

Since t_1 and t_2 are minimal upper bounds of C, there are $c_1, c_2 \in C$ with $s_1 < c_1 \leq t_1$ and $s_2 < c_2 \leq t_2$. Otherwise, s_1 and s_2 would be an upper bound of C strictly below t_1 and t_2, respectively.

As C is a chain, assume without loss of generality that $c_1 \leq c_2$. Then $s_1 < c_1 \leq c_2 \leq t_1$. Hence, $c_2 \in (s_1, t_1] \subset U$. Also $s_2 < c_2 \leq t_2$, so $c_2 \in (s_2, t_2] \subset V$. Therefore $c_2 \in U \cap V$, which contradicts $U \cap V = \emptyset$.

Hence $\mathrm{MUB}(C)$ is a singleton, say $\{m\}$. Since $\mathrm{MUB}(C)$ is valent with respect to $\mathrm{UB}(C)$, every upper bound $u \in \mathrm{UB}(C)$ is comparable with m, and minimality of m gives $m \leq u$. Therefore m is the least upper bound of C. □

Theorem 3.7. *Let T be a MUB-valent pseudotree. Then the following are equivalent for T:*

(i) *Hausdorff,*
(ii) *Dedekind complete,*
(iii) *zero-dimensional,*
(iv) *Tychonoff.*

Proof. The implication (i) $\Rightarrow$ (ii) is Lemma 3.6. The implications (ii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iii), and (ii) $\Rightarrow$ (iv) are Theorem 3.5. The implication (iv) $\Rightarrow$ (i) is immediate.

Finally, suppose that T is zero-dimensional. Let $t_1, t_2 \in T$ be distinct. Since the upper-limit topology is T_0 by Proposition 3.3, there is an open set O that separates the points. Assume $t_1 \in O$ and $t_2 \notin O$. By zero-dimensionality, choose a clopen neighborhood $U \subset O$ of t_1. Let $V = U^c$. Then V is an open neighborhood of the other point, and $U \cap V = \emptyset$. Thus T is Hausdorff. □

4. Limit Structure and Homogeneous Local Bases

The previous sections show that, if a pseudotree is immediately branching, MUB-valent, and Hausdorff, then it is locally tree-like. We now describe the local limit structure.

Since a pseudotree need not be a tree, the height of a point is not available. We bypass this by defining the relative height and relative cofinality of a point.

For the rest of the paper, T is a Hausdorff, immediately branching, MUB-valent pseudotree. By Theorem 3.7, T is also Tychonoff and Dedekind complete, and the basic half-open intervals, $(s, t]$, are clopen. Moreover, for each $t \in T$, the set $t^{\geq}$ is a tree by Theorem 2.43.

4.1. Limit and Immediate Successor Points.

We give basic definitions and prove results relating topologically isolated and limit points to order-theoretic limits in immediately branching pseudotrees.

Definition 4.1. Let t be a non-minimal point and $s < t$. Since $s^{\geq}$ is a tree, the interval $t^{<} \cap s^{\geq} = [s, t)$ is well-ordered. Define the *relative order* *type of t with respect to s* to be

$$h_s(t) := \operatorname{otp}([s, t)).$$

Definition 4.2. Let t be a non-minimal point of T.

(i) We say t has *relative limit height* if there exists $s < t$ such that $h_s(t)$ is a non-zero limit ordinal.

(ii) We say t has *relative successor height* if there exists $s < t$ such that $h_s(t)$ is a successor ordinal.

Proposition 4.3. *If a non-minimal point t has relative successor height, then there exists $s < t$ such that $h_s(t) = 1$.*

Proof. If t has relative successor height, then there is an $s' < t$ such that $h_{s'}(t) = \alpha + 1$ for some ordinal α. Enumerate $[s', t)$ in increasing order as $\langle t_\beta : \beta < \alpha + 1\rangle$. Since t_α is the last element of $[s', t)$, we have $[t_\alpha, t) = \{t_\alpha\}$, so $h_{t_\alpha}(t) = 1$. $\square$

Proposition 4.4. *A non-minimal point t has relative successor height if and only if t is an immediate successor.*

Proof. If t has relative successor height, then there exists $s < t$ such that $h_s(t) = 1$ by Proposition 4.3. Hence, $\operatorname{otp}([s, t)) = 1$ and $|[s, t)| = 1$. Since $s \in [s, t)$, $[s, t) = \{s\}$. Thus, there is no $t' \in T$ such that $s < t' < t$, and $t \in s^+$. Therefore t is an immediate successor.

If t is an immediate successor, let t^- be the immediate predecessor. Since $t \in (t^-)^+$, $t^- < t$ and there is no $t' \in T$ such that $t^- < t' < t$. Then, $[t^-, t) = (t^-)^{\geq} \cap t^{<} = \{t^-\}$. Since $|\{t^-\}| = 1$, $\operatorname{otp}([t^-, t)) = 1$. Therefore $h_{t^-}(t)$ is a successor ordinal. $\square$

Proposition 4.5. *If a non-minimal point t has relative limit height with respect to some $s \in T$ with $s < t$, then it has relative limit height with respect to every $s' < t$.*

Proof. We use the ordinal-arithmetic fact that, for ordinals α and β, the ordinal $\alpha + \beta$ is a limit ordinal whenever β is a non-zero limit ordinal.

Suppose that t has relative limit height with respect to $s < t$. Take any $s' < t$. If $s' = s$, then this is immediate.

If $s' < s$, then

$$h_{s'}(t) = h_{s'}(s) + h_s(t),$$

so $h_{s'}(t)$ is a non-zero limit ordinal. If $s < s' < t$, then

$$h_s(t) = h_s(s') + h_{s'}(t).$$

If $h_{s'}(t)$ were a successor ordinal, then $h_s(t)$ would also be a successor ordinal, a contradiction. Since $s' < t$, $h_{s'}(t) \neq 0$. Therefore $h_{s'}(t)$ is a non-zero limit ordinal. $\square$

Definition 4.6. A point $t \in T$ is a *topological limit point* if every neighborhood of t contains a point other than t.

Proposition 4.7. *A non-minimal point $t \in T$ has relative limit height if and only if t is a topological limit point.*

Proof. Suppose that t has relative limit height with respect to some $s < t$. In the tree $s^{\geq}$, the set $[s, t)$ has order type α for some non-zero limit ordinal α. Enumerate $[s, t)$ increasingly as $\langle t_\beta : \beta < \alpha \rangle$.

Let $(s', t]$ be a basic neighborhood of t. If $s' < s$, then $s \in (s', t]$. If $s \leq s' < t$, then $s' = t_\gamma$ for some $\gamma < \alpha$, and since α is a limit ordinal, $t_{\gamma+1} \in (s', t]$. Thus t is a topological limit point.

Conversely, suppose that t is a topological limit point. Fix $s < t$ and let $\alpha = h_s(t)$. Enumerate $[s, t)$ increasingly as $\langle t_\beta : \beta < \alpha \rangle$.

If $h_s(t)$ were a successor ordinal, then $\alpha = \gamma + 1$. Thus $(t_\gamma, t] = \{t\}$. This contradicts that t is a topological limit point. Therefore $h_s(t)$ is a non-zero limit ordinal. □

Corollary 4.8. *Every point of T is exactly*

(i) *minimal,*
(ii) *an immediate successor, or*
(iii) *a limit point.*

Proof. Minimal points are never immediate successors as they have no immediate predecessor, and they are not limit points as minimal points are isolated by Proposition 3.2.

If t is non-minimal, then there is some $s < t$. Hence, $h_s(t)$ is either a successor or non-zero limit ordinal. Proposition 4.4 shows that t is an immediate successor if $h_s(t)$ is a successor, and Proposition 4.7 shows that t is a limit point if $h_s(t)$ is a non-zero limit. These are mutually exclusive as ordinals cannot be both successors and limits. □

Remark 4.9. We now simply say points are *limit points* or *immediate successors* with no reference to their relative height. Together, Corollary 4.8 along with Propositions 4.7 and 4.4 leave no ambiguity.

4.2. **Relative Cofinality.** We define a cardinal invariant for each limit point called the *relative cofinality.*

Proposition 4.10. *Let t be a limit point and $s, s' < t$. Then* $\mathrm{cf}(h_s(t)) = \mathrm{cf}(h_{s'}(t))$.

Proof. We use Proposition 4.5 and the fact that if β is a non-zero limit ordinal, then $\mathrm{cf}(\alpha + \beta) = \mathrm{cf}(\beta)$ for every ordinal α.

Fix any $s, s' < t$. The case $s = s'$ is immediate. Say $s < s' < t$. Then

$$h_s(t) = h_s(s') + h_{s'}(t).$$

Since $h_{s'}(t)$ is also a non-zero limit ordinal,

$$\mathrm{cf}(h_s(t)) = \mathrm{cf}(h_s(s') + h_{s'}(t)) = \mathrm{cf}(h_{s'}(t))$$

as desired. The argument is symmetric if $s' < s < t$. □

Definition 4.11. Let t be a limit point. We define the *relative cofinality of* t, denoted $\mathrm{cf}_T(t)$, to be $\mathrm{cf}(h_s(t))$ for any $s < t$. If t is an immediate successor or is minimal, we set $\mathrm{cf}_T(t) = 1$.

Remark 4.12. When t is a limit point, $\mathrm{cf}_T(t)$ is a regular cardinal. Hence it is an infinite limit ordinal. Moreover, if $\langle t_\mu : \mu < \mathrm{cf}_T(t)\rangle$ is a strictly increasing cofinal non-empty sequence in $t^<$, then any subsequence of cardinality $< \mathrm{cf}_T(t)$ has supremum strictly below t by Dedekind completeness.

Definition 4.13. Let $\langle t_\mu : \mu < \alpha\rangle$ be a sequence of points indexed by a non-zero limit ordinal α. We write $t_\mu \to t$ to mean that for every neighborhood U of t, there exists $\gamma < \alpha$ such that $\{t_\mu : \gamma < \mu < \alpha\} \subset U$.

Proposition 4.14. *If t is a limit point, then there exists a strictly increasing sequence $\langle t_\mu : \mu < \mathrm{cf}_T(t)\rangle$ with $t_\mu < t$ for every $\mu < \mathrm{cf}_T(t)$ such that $t_\mu \to t$. If t is an immediate successor or is minimal, then t is isolated.*

Proof. Let $s < t$, and consider the tree $s^{\geq}$. Let $\langle \alpha_\mu : \mu < \mathrm{cf}_T(t)\rangle$ be any strictly increasing and cofinal sequence in $h_s(t)$ with each $\alpha_\mu < h_s(t)$. For each $\mu < \mathrm{cf}_T(t)$, let t_μ be the α_μ-th element of $[s, t)$ in the tree $s^{\geq}$.

To see that $t_\mu \to t$, let $(t', t]$ be a basic open neighborhood of t. Assume $t' \geq s$. By cofinality, there exists $\gamma < \mathrm{cf}_T(t)$ such that $t_\gamma > t'$. Since $\langle t_\mu : \mu < \mathrm{cf}_T(t)\rangle$ is strictly increasing, $\{t_\mu : \gamma < \mu < \mathrm{cf}_T(t)\} \subset (t', t]$.

If $t' < s$, then $\{t_\mu : 0 < \mu < \mathrm{cf}_T(t)\} \subset (t', t]$. In either case, it follows that $t_\mu \to t$.

If t is an immediate successor, then $(t^-, t] = \{t\}$. If t is minimal, then $t^{\leq} = \{t\}$. Thus immediate successors and minimal points are isolated. □

Proposition 4.15. *Let t be a limit point, and let $\langle t_\mu : \mu < \mathrm{cf}_T(t)\rangle$ be a strictly increasing sequence with $t_\mu < t$ for every $\mu < \mathrm{cf}_T(t)$ and $t_\mu \to t$. Then $\{(t_\mu, t] : \mu < \mathrm{cf}_T(t)\}$ is a local base at t.*

Proof. For each $s < t$ and each open neighborhood $(s, t]$, convergence gives some $\mu < \mathrm{cf}_T(t)$ with $t_\mu > s$. Hence $(t_\mu, t] \subset (s, t]$. □

4.3. **Limiting Maps and Monotone Clopen Bases.** We define the tools needed to discuss clopen local bases with interval-like behavior.

Definition 4.16. Let $t \in T$. A map $\phi_t : \mathrm{cf}_T(t) \to T$ is called *limiting* if

(i) ϕ_t is strictly increasing and cofinal in $t^<$ when t is a limit point,

(ii) $\phi_t(0) = t^-$ when t is an immediate successor, or
(iii) $\phi_t(0) = t$ when t is minimal.

Definition 4.17. A collection of maps $\mathcal{M} = \{\phi_t : t \in T\}$ is *uniform limiting* if ϕ_t is limiting for each t.

Definition 4.18. An indexed clopen local base $\mathcal{A}_t = \{A_{\mu,t} : \mu < \mathrm{cf}_T(t)\}$ at a point t is called *monotone* if

(i) $A_{\mu,t} \subset t^{\leq}$ for every $\mu < \mathrm{cf}_T(t)$, and
(ii) $A_{\nu,t} \subsetneq A_{\mu,t}$ for every $\mu < \nu < \mathrm{cf}_T(t)$.

A collection of clopen local bases $\mathcal{A} = \{\mathcal{A}_t : t \in T\}$ is *monotone* if each $\mathcal{A}_t$ is monotone.

Remark 4.19. For an isolated point t, since $\mathrm{cf}_T(t) = 1$, a monotone local base $\mathcal{A}_t$ has a single neighborhood, namely $A_{0,t} = \{t\}$.

4.4. **Homogeneous Pairs.** Homogeneity requires that the chosen clopen neighborhoods behave like the canonical half-open intervals.

Definition 4.20. Let $\mathcal{M}$ be a uniform limiting collection, and let $\mathcal{A}$ be a monotone collection of clopen local bases. We call the pair $(\mathcal{M}, \mathcal{A})$ *homogeneous* if, for each limit point t and $\mu < \nu < \mathrm{cf}_T(t)$, the following chain of inclusions holds:

$$A_{\nu,t} \subset (\phi_t(\nu), t] \subsetneq A_{\mu,t} \subset (\phi_t(\mu), t].$$

Remark 4.21. If $(\mathcal{M}, \mathcal{A})$ is homogeneous and t is a limit point, then for each $\mu < \mathrm{cf}_T(t)$, the inclusion

$$A_{\mu,t} \subset (\phi_t(\mu), t]$$

holds because we may apply homogeneity with $\mu < \mu + 1 < \mathrm{cf}_T(t)$.

Proposition 4.22. *Let $(\mathcal{M}, \mathcal{A})$ be a homogeneous pair, and let t be a limit point. For any fixed $\mu < \mathrm{cf}_T(t)$, there exists a finite set $E_{\mu,t} \in [T]^{<\omega}$ such that for all $\nu < \mathrm{cf}_T(t)$ with $\nu \neq \mu$,*

$$E_{\mu,t} \cap (A_{\mu,t} \Delta A_{\nu,t}) \neq \emptyset.$$

Proof. Fix a limit point t and $\mu < \mathrm{cf}_T(t)$.

If $\mu = 0$, then $A_{0,t} \setminus A_{1,t} \neq \emptyset$ by monotonicity. Let $p_{0,t}$ be any element of that set difference, and let $E_{0,t} = \{p_{0,t}\}$. By monotonicity, for all $\nu > 0$, $p_{0,t} \notin A_{\nu,t}$.

If $\mu = \delta + 1$ is a successor ordinal, then $A_{\mu,t} \setminus A_{\mu+1,t}$ and $A_{\delta,t} \setminus A_{\mu,t}$ are non-empty by monotonicity. Choose $p_{\mu,t}$ from the first set difference, and choose $q_{\mu,t}$ from the second set difference. Let $E_{\mu,t} = \{p_{\mu,t}, q_{\mu,t}\}$. If $\mu < \nu$, then $p_{\mu,t} \in A_{\mu,t} \setminus A_{\nu,t}$. If $\nu < \mu$, then $q_{\mu,t} \in A_{\nu,t} \setminus A_{\mu,t}$.

Let μ be a non-zero limit ordinal. Monotonicity gives $A_{\mu,t} \setminus A_{\mu+1,t} \neq \emptyset$, so choose $p_{\mu,t} \in A_{\mu,t} \setminus A_{\mu+1,t}$. Set $q_{\mu,t} = \phi_t(\mu)$, and let $E_{\mu,t} = \{p_{\mu,t}, q_{\mu,t}\}$. If $\mu < \nu$, then $p_{\mu,t} \in A_{\mu,t} \setminus A_{\nu,t}$. If $\nu < \mu$, choose η with $\nu < \eta < \mu$. Homogeneity with $\nu < \eta$ gives $(\phi_t(\eta), t] \subsetneq A_{\nu,t}$, so $q_{\mu,t} \in A_{\nu,t}$. However, $A_{\mu,t} \subset (\phi_t(\mu), t]$, so $q_{\mu,t} \notin A_{\mu,t}$. □

5. Generators and the Classification Theorem

We study binary generators on immediately branching, MUB-valent, Hausdorff pseudotrees. The existence of such generators is determined by the ability of each neighborhood to detect all other non-disjoint neighborhoods via a finite set of points. The main theorem shows that the existence of a homogeneous pair satisfying a finite separation property is precisely the finite-detection property.

5.1. Function-Space Preliminaries.

We review definitions from $C_p(X)$ theory.

Definition 5.1. Let X be a Tychonoff space. A collection $\mathcal{G}$ of real-valued continuous functions on X is a *generator* for X if, for every $x \in X$ and every closed set $K \subset X$ such that $x \notin K$, there exists $g \in \mathcal{G}$ such that $g(x) \notin \overline{g(K)}$.

Definition 5.2. Let $C_p(X)$ denote the set of all continuous real-valued functions on a Tychonoff space X, endowed with the topology of pointwise convergence. Basic open neighborhoods of $f \in C_p(X)$ are the sets of the form

$$U(f, F, \epsilon) = \{g \in C_p(X) : \forall x \in F, |f(x) - g(x)| < \epsilon\}$$

where $\epsilon > 0$ and $F \in [X]^{<\omega}$.

Definition 5.3. A generator $\mathcal{G}$ is called *discrete* if it is a discrete subspace of $C_p(X)$.

Definition 5.4. A generator $\mathcal{G}$ is a $(0,1)$*-generator* if, for every $x \in X$ and every closed set $K \subset X$ such that $x \notin K$, there is $g \in \mathcal{G}$ such that $g(x) = 1$ and $g(K) \subset \{0\}$.

Definition 5.5. A discrete $(0,1)$-generator $\mathcal{G}$ is called a *binary generator* if $\operatorname{ran}(g) \subset \{0,1\}$ for every $g \in \mathcal{G}$.

Proposition 5.6. *A continuous real-valued function $f : X \to \mathbb{R}$ with range contained in $\{0,1\}$ is of the form χ_A for some clopen set $A \subset X$. Conversely, χ_A is continuous whenever A is clopen.*

Proof. Let $f : X \to \mathbb{R}$ be continuous with $\operatorname{ran}(f) \subset \{0,1\}$. Fix $0 < \epsilon < \frac{1}{2}$, and let $B_\epsilon(1) \subset \mathbb{R}$ be the ball of radius ϵ around 1. Then $A =$

$f^{-1}(B_\epsilon(1)) = f^{-1}(\{1\})$. Hence A is clopen. Moreover, if $x \in A$, then $f(x) = 1$, and if $x \in X \setminus A$, then $f(x) = 0$. Thus $f = \chi_A$.

Conversely, if $U \subset \mathbb{R}$ is open, then $\chi_A^{-1}(U) \in \{\emptyset, X, A, A^c\}$. Hence, the inverse image of any open set is open since A is clopen. □

5.2. **The Finite Separation Condition and Main Theorem.** We now define the finite separation property for homogeneous pairs. Since discreteness of a subset $\mathcal{G} \subset C_p(T)$ is a finite separation condition for each member $g \in \mathcal{G}$, each characteristic function must be isolated from the others by its values on finitely many points of T. We translate that condition to a combinatorial property on limiting maps and monotone clopen local bases.

Definition 5.7. Let $(\mathcal{M}, \mathcal{A})$ be a homogeneous pair. The pair *finitely separates at t* if, for every $\mu < \mathrm{cf}_T(t)$, there exists a finite set $F_{\mu,t} \in [T]^{<\omega}$ such that, for every $t' > t$ and every $\nu < \mathrm{cf}_T(t')$, if $t \in A_{\nu,t'}$, then

$$(A_{\mu,t} \Delta A_{\nu,t'}) \cap F_{\mu,t} \neq \emptyset.$$

Separation among neighborhoods at the same point is handled by Proposition 4.22.

Proposition 5.8. *Let $(\mathcal{M}, \mathcal{A})$ be a homogeneous pair. The mapping*

$$(t, \mu) \mapsto \chi_{A_{\mu,t}}$$

from $\{(t, \mu) : t \in T, \mu < \mathrm{cf}_T(t)\}$ to $C_p(T)$ is injective.

Hence, distinct pairs $(t, \mu) \neq (t', \nu)$ yield distinct characteristic functions $\chi_{A_{\mu,t}} \neq \chi_{A_{\nu,t'}}$.

Proof. This mapping is well-defined since each $A_{\mu,t}$ is clopen.

If $t < t'$, then $t' \in A_{\nu,t'}$ but $t' \notin A_{\mu,t}$, since $A_{\mu,t} \subset t^{\leq}$. Hence $1 = \chi_{A_{\nu,t'}}(t') \neq \chi_{A_{\mu,t}}(t') = 0$.

The case $t' < t$ is symmetric.

If $t = t'$ and $\mu \neq \nu$ with both $\mu, \nu < \mathrm{cf}_T(t)$, then monotonicity guarantees that either $A_{\mu,t} \setminus A_{\nu,t} \neq \emptyset$ or $A_{\nu,t} \setminus A_{\mu,t} \neq \emptyset$. In either case, there exists a point s in the set difference such that $\chi_{A_{\mu,t}}(s) \neq \chi_{A_{\nu,t}}(s)$.

If t and t' are incomparable, then $t \notin A_{\nu,t'}$ and $t' \notin A_{\mu,t}$. Then $1 = \chi_{A_{\mu,t}}(t) \neq \chi_{A_{\nu,t'}}(t) = 0$.

Hence, distinct pairs of points $(t, \mu) \neq (t', \nu)$ give distinct characteristic functions $\chi_{A_{\mu,t}}$ and $\chi_{A_{\nu,t'}}$. □

The final theorem shows that for locally tree-like pseudotrees, the existence of a binary generator is equivalent to a combinatorial condition on clopen local bases.

Theorem 5.9. *Let T be a Hausdorff, immediately branching, MUB-valent pseudotree. Then the following are equivalent:*

(i) *There exists a homogeneous pair $(\mathcal{M}, \mathcal{A})$ that finitely separates every point $t \in T$.*
(ii) *T admits a binary generator.*

5.3. **Sufficiency.** We first show that a homogeneous pair with finite separation yields a binary generator by taking characteristic functions of each member of every clopen local base.

Assume T is a Hausdorff, immediately branching, MUB-valent pseudotree with a homogeneous pair $(\mathcal{M}, \mathcal{A})$ that finitely separates at every point.

Lemma 5.10. *Let $(\mathcal{M}, \mathcal{A})$ be a homogeneous pair that finitely separates at every point. Let*

$$\mathcal{G} = \{g_{\mu,t} : t \in T, \mu < \mathrm{cf}_T(t)\}$$

where $g_{\mu,t} = \chi_{A_{\mu,t}}$. Then $\mathcal{G}$ is a $(0,1)$-generator.

Proof. Fix $t \in T$ and a closed set $K \subset T$ with $t \notin K$. Since $T \setminus K$ is a neighborhood of t and $\mathcal{A}_t$ is a local base of clopen sets, there exists $\mu < \mathrm{cf}_T(t)$ such that $A_{\mu,t} \cap K = \emptyset$. Hence $g_{\mu,t}(t) = 1$ and $g_{\mu,t}(K) \subset \{0\}$. Each $g_{\mu,t}$ is continuous by Proposition 5.6. □

Throughout the next two lemmas, for each $t \in T$ and $\mu < \mathrm{cf}_T(t)$, let $F_{\mu,t}$ be the set guaranteed by finite separation.

Lemma 5.11. *If t is isolated in T, then $g_{0,t}$ is isolated in $\mathcal{G}$.*

Proof. Since t is isolated, $A_{0,t} = \{t\}$ by Remark 4.19. Set

$$U_{0,t} = U\left(g_{0,t}, \{t\} \cup F_{0,t}, \frac{1}{2}\right).$$

We show that $U_{0,t} \cap \mathcal{G} = \{g_{0,t}\}$.

Let $g_{\nu,t'} \in \mathcal{G}$ be distinct from $g_{0,t}$. If t' is isolated, then $\nu = 0$ and $g_{\nu,t'} = g_{0,t'} = \chi_{\{t'\}}$, and

$$|g_{0,t}(t) - g_{0,t'}(t)| = 1.$$

If t' is a limit point and $t \notin A_{\nu,t'}$, then

$$|g_{0,t}(t) - g_{\nu,t'}(t)| = 1.$$

Finally, suppose t' is a limit point and $t \in A_{\nu,t'}$. Since $t \in A_{\nu,t'} \subset t'^{\leq}$, we have $t < t'$. Hence, finite separation at t gives a point $p \in F_{0,t} \cap (A_{0,t} \Delta A_{\nu,t'})$. Therefore

$$|g_{0,t}(p) - g_{\nu,t'}(p)| = 1,$$

and $U_{0,t} \cap \mathcal{G} = \{g_{0,t}\}$. □

Lemma 5.12. *If t is a limit point in T, then $g_{\mu,t}$ is isolated in $\mathcal{G}$ for each $\mu < \mathrm{cf}_T(t)$.*

Proof. Fix a limit point t and $\mu < \mathrm{cf}_T(t)$. Let $E_{\mu,t}$ be the finite set supplied by Proposition 4.22, and set

$$H_{\mu,t} = \{t\} \cup F_{\mu,t} \cup E_{\mu,t}.$$

Consider $U_{\mu,t} = U(g_{\mu,t}, H_{\mu,t}, \frac{1}{2})$.

Let $g_{\nu,t'} \in \mathcal{G}$ be distinct from $g_{\mu,t}$. If t' is isolated, then $g_{\nu,t'} = g_{0,t'} = \chi_{\{t'\}}$. Hence,

$$|g_{\mu,t}(t) - g_{0,t'}(t)| = 1.$$

Suppose t' is a limit point and $t \neq t'$. If $t \notin A_{\nu,t'}$, then

$$|g_{\mu,t}(t) - g_{\nu,t'}(t)| = 1.$$

If $t \in A_{\nu,t'}$, then $t < t'$. Hence, finite separation at t gives $f \in F_{\mu,t}$ such that

$$|g_{\mu,t}(f) - g_{\nu,t'}(f)| = 1.$$

Finally, if $t = t'$, then $\nu \neq \mu$, and Proposition 4.22 gives a point $e \in E_{\mu,t}$ where

$$|g_{\mu,t}(e) - g_{\nu,t}(e)| = 1.$$

Thus $U_{\mu,t} \cap \mathcal{G} = \{g_{\mu,t}\}$. □

Lemma 5.13. *$\mathcal{G}$ is a binary generator.*

Proof. By Lemma 5.10, $\mathcal{G}$ is a $(0,1)$-generator. Lemmas 5.11 and 5.12 show that every member of $\mathcal{G}$ is isolated in $\mathcal{G}$. Hence $\mathcal{G}$ is discrete. By construction, every member of $\mathcal{G}$ is $\{0,1\}$-valued. □

5.4. **Necessity.** Since the members of a binary generator are characteristic functions of clopen sets, the generator provides the clopen local bases. A transfinite recursion refines these bases into monotone clopen local bases and limiting maps. Discreteness of the generator guarantees finite separation.

Assume now that $\mathcal{G}$ is a binary generator for an immediately branching, MUB-valent, Hausdorff pseudotree T.

Lemma 5.14. *For each limit point $t \in T$ and each strictly increasing cofinal sequence $\langle t_\mu : \mu < \mathrm{cf}_T(t)\rangle \subset t^{<}$, the generator $\mathcal{G}$ contains the characteristic functions of a clopen local base $\mathcal{B}_t = \{B_{\mu,t} : \mu < \mathrm{cf}_T(t)\}$ at t such that $B_{\mu,t} \subset (t_\mu, t]$.*

Proof. Fix a strictly increasing cofinal sequence $\langle t_\mu : \mu < \mathrm{cf}_T(t)\rangle$. For each $\mu < \mathrm{cf}_T(t)$, the interval $(t_\mu, t]$ is clopen by Theorem 3.5. Hence $(t_\mu, t]^c$ is closed and does not contain t. Since $\mathcal{G}$ is a $(0,1)$-generator, there exists $h_{\mu,t} \in \mathcal{G}$ such that $h_{\mu,t}(t) = 1$ and $h_{\mu,t}((t_\mu, t]^c) \subset \{0\}$.

Since $\mathcal{G}$ is binary, $h_{\mu,t}$ has range contained in $\{0,1\}$. Set $B_{\mu,t} = h_{\mu,t}^{-1}(\{1\})$. Then $t \in B_{\mu,t}$, $B_{\mu,t}$ is clopen, $B_{\mu,t} \subset (t_\mu, t]$, and $h_{\mu,t} = \chi_{B_{\mu,t}} \in \mathcal{G}$.

The family $\mathcal{B}_t$ is a local base at t. For any basic neighborhood $(s,t]$, since the sequence $\langle t_\mu : \mu < \mathrm{cf}_T(t)\rangle$ is cofinal, there exists $\nu < \mathrm{cf}_T(t)$ such that $s < t_\nu < t$. Then $t \in B_{\nu,t}$ and

$$B_{\nu,t} \subset (t_\nu, t] \subset (s,t].$$

$\square$

Lemma 5.15. *In the setting of Lemma 5.14, there exists a cofinal strictly increasing sequence $\langle i_\mu : \mu < \mathrm{cf}_T(t)\rangle \subset \mathrm{cf}_T(t)$, a monotone clopen local base $\mathcal{A}_t = \{B_{i_\mu,t} : \mu < \mathrm{cf}_T(t)\}$, and a limiting map ϕ_t.*

Proof. Let $\kappa = \mathrm{cf}_T(t)$. Given a strictly increasing cofinal sequence $\langle t_\mu : \mu < \kappa\rangle$, let $\mathcal{B}_t = \{B_{\mu,t} : \mu < \kappa\}$ be the clopen local base from Lemma 5.14. We construct a subsequence of indices $\langle i_\mu : \mu < \kappa\rangle$ in κ recursively.

For $\mu = 0$, set $i_0 = 0$. Assume i_μ has been defined. Since $B_{i_\mu,t}$ is an open neighborhood of t, there exists $s < t$ such that $(s,t] \subset B_{i_\mu,t}$. As $\langle t_\mu : \mu < \kappa\rangle$ is cofinal, there exists $\eta < \kappa$ such that $t_\eta > s$. Hence,

$$(t_\eta, t] \subsetneq (s,t] \subset B_{i_\mu,t}.$$

Set $i_{\mu+1} = \max\{i_\mu, \eta\} + 1$. Note that $i_{\mu+1} > i_\mu$, and if $i_\mu \geq \mu$, then $i_{\mu+1} \geq \mu + 1$. The inclusion

$$(t_{i_{\mu+1}}, t] \subsetneq B_{i_\mu,t}$$

follows from $(t_{i_{\mu+1}}, t] \subsetneq (t_\eta, t] \subset B_{i_\mu,t}$ since $t_\eta < t_{i_{\mu+1}}$ because $i_{\mu+1} > \eta$ and $\langle t_\mu : \mu < \kappa\rangle$ is strictly increasing.

Assume i_μ has been defined for all $\mu < \delta$, where $\delta < \kappa$ is a non-zero limit ordinal. Since $\delta < \kappa$ and κ is a regular cardinal, $\gamma := \sup\{i_\mu : \mu < \delta\} < \kappa$. Set $i_\delta = \max\{\gamma + 1, \delta\}$. For each $\mu < \delta$, $i_\delta > i_\mu$ since $i_\delta \geq \gamma + 1 > \gamma \geq i_\mu$, and $i_\delta \geq \delta$.

The sequence $\langle i_\mu : \mu < \kappa\rangle$ is strictly increasing and satisfies $i_\mu \geq \mu$. Hence $\langle i_\mu : \mu < \kappa\rangle$ is cofinal in κ.

Consider the collection $\{B_{i_\mu,t} : \mu < \kappa\}$. For every $\mu < \nu < \kappa$, the following chain of inclusions holds:

$$\begin{aligned} (t_{i_{\nu+1}}, t] \subsetneq B_{i_\nu,t} &\subset (t_{i_\nu}, t] \\ &\subset (t_{i_{\mu+1}}, t] \subsetneq B_{i_\mu,t} \subset (t_{i_\mu}, t]. \end{aligned} \tag{5.1}$$

The first and fourth inclusions follow from the construction, the second and fifth follow from Lemma 5.14, and the third follows from $i_\mu < i_{\mu+1} \leq i_\nu$. Hence, $B_{i_\nu,t} \subsetneq B_{i_\mu,t}$.

The collection is a local base at t. For any neighborhood $(s,t]$, there exists $\mu<\kappa$ with $s<t_\mu$. Hence,

$$B_{i_\mu,t}\subset(t_{i_\mu},t]\subset(t_\mu,t]\subset(s,t].$$

Define $A_{\mu,t}=B_{i_\mu,t}$. Then $\mathcal{A}_t=\{A_{\mu,t}:\mu<\kappa\}$ is a monotone clopen local base at t.

Define $\phi_t(\mu)=t_{i_\mu}$. Since $\langle i_\mu:\mu<\kappa\rangle$ is strictly increasing and cofinal in κ, and $\langle t_\mu:\mu<\kappa\rangle$ is strictly increasing and cofinal in $t^<$, $\langle\phi_t(\mu):\mu<\kappa\rangle$ is strictly increasing and cofinal in $t^<$. Hence ϕ_t is limiting. □

Lemma 5.16. *For each limit point t, the monotone clopen local base and limiting map from Lemma 5.15 satisfy*

$$A_{\nu,t}\subset(\phi_t(\nu),t]\subsetneq A_{\mu,t}\subset(\phi_t(\mu),t]$$

whenever $\mu<\nu<\mathrm{cf}_T(t)$.

Proof. This follows from equation (5.1) in the proof of Lemma 5.15. □

Lemma 5.17. *$\mathcal{G}$ yields a homogeneous pair $(\mathcal{M},\mathcal{A})$.*

Proof. For each limit point $t\in T$, fix a strictly increasing cofinal sequence $\langle t_\mu:\mu<\mathrm{cf}_T(t)\rangle$, and let $\mathcal{A}_t$ and ϕ_t be obtained from Lemma 5.15. For each isolated point t, set $A_{0,t}=\{t\}$ and let $\mathcal{A}_t=\{A_{0,t}\}$.

By Corollary 4.8 and Proposition 4.14, every isolated point is either minimal or an immediate successor. If t is minimal, set $\phi_t(0)=t$. If t is an immediate successor, set $\phi_t(0)=t^-$.

Let $\mathcal{M}=\{\phi_t:t\in T\}$ and $\mathcal{A}=\{\mathcal{A}_t:t\in T\}$. For limit points, homogeneity follows from Lemma 5.16. Thus $(\mathcal{M},\mathcal{A})$ is homogeneous. □

Lemma 5.18. *For each limit point t and $\mu<\mathrm{cf}_T(t)$, $\chi_{A_{\mu,t}}\in\mathcal{G}$.*

Proof. The result follows from Lemma 5.14. Since $A_{\mu,t}=B_{i_\mu,t}$ and $\chi_{B_{i_\mu,t}}\in\mathcal{G}$, $\chi_{A_{\mu,t}}\in\mathcal{G}$. □

Lemma 5.19. *For each isolated point $t\in T$, the function $\chi_{\{t\}}$ belongs to $\mathcal{G}$.*

Proof. If t is isolated, then $\{t\}^c$ is closed since $\{t\}$ is open. By the $(0,1)$-generator property, there is $g_t\in\mathcal{G}$ such that $g_t(t)=1$ and $g_t(\{t\}^c)\subset\{0\}$. Hence $g_t=\chi_{\{t\}}$. □

Lemma 5.20. *The homogeneous pair $(\mathcal{M},\mathcal{A})$ determined by $\mathcal{G}$ finitely separates at every point $t\in T$.*

Proof. Fix $t\in T$ and $\mu<\mathrm{cf}_T(t)$. If t is a limit point, Lemma 5.18 gives $\chi_{A_{\mu,t}}\in\mathcal{G}$. If t is isolated, Lemma 5.19 also gives $\chi_{A_{\mu,t}}\in\mathcal{G}$. Since $\mathcal{G}$ is discrete, there exist $\epsilon>0$ and a finite set $F\in[T]^{<\omega}$ such that

$$U(\chi_{A_{\mu,t}},F,\epsilon)\cap\mathcal{G}=\{\chi_{A_{\mu,t}}\}.$$

We may assume $0 < \epsilon \leq \frac{1}{2}$. Let $F_{\mu,t} = F$.

Suppose that $t' > t$, $\nu < \mathrm{cf}_T(t')$, and $t \in A_{\nu,t'}$. Then t' cannot be isolated because this would imply $t \in A_{\nu,t'} = A_{0,t'} = \{t'\}$, contradicting $t' > t$. Hence t' is a limit point and Lemma 5.18 gives $\chi_{A_{\nu,t'}} \in \mathcal{G}$.

Moreover, since $(t, \mu) \neq (t', \nu)$, Proposition 5.8 gives $\chi_{A_{\nu,t'}} \neq \chi_{A_{\mu,t}}$. Hence $\chi_{A_{\nu,t'}} \notin U(\chi_{A_{\mu,t}}, F, \epsilon)$. Thus there is $f \in F$ such that

$$|\chi_{A_{\mu,t}}(f) - \chi_{A_{\nu,t'}}(f)| \geq \epsilon.$$

Both functions are characteristic functions. Hence $f \in A_{\mu,t} \Delta A_{\nu,t'}$ and therefore

$$F_{\mu,t} \cap (A_{\mu,t} \Delta A_{\nu,t'}) \neq \emptyset.$$

□

Proof of Theorem 5.9. The direction (i) ⇒ (ii) is given by Lemma 5.13. The direction (ii) ⇒ (i) is given by Lemmas 5.17 and 5.20. □

5.5. **Corollaries.** We now show that locally tree-like pseudotrees with finite valence admit binary generators. Since ordinals are well-ordered finitely branching trees, the ordinal result in [4] follows as a special case.

Corollary 5.21. *Let T be an immediately branching, MUB-valent, Hausdorff pseudotree. If $\lambda_t < \omega$ for every $t \in T$, then T has a binary generator.*

Proof. For each limit point $t \in T$, choose a strictly increasing cofinal sequence $\langle t_\mu : \mu < \mathrm{cf}_T(t)\rangle$, and set $A_{\mu,t} = (t_\mu, t]$ and $\phi_t(\mu) = t_\mu$. If t is an immediate successor, set $A_{0,t} = \{t\}$ and $\phi_t(0) = t^-$. If t is minimal, set $A_{0,t} = \{t\}$ and $\phi_t(0) = t$.

Let $\mathcal{M}$ and $\mathcal{A}$ be the corresponding collections of maps and bases. Then $(\mathcal{M}, \mathcal{A})$ is homogeneous.

We show that the pair finitely separates. Fix $t \in T$ and $\mu < \mathrm{cf}_T(t)$, and set $F_{\mu,t} = t^+$. If t is maximal, then $t^+ = \emptyset$, and the finite-separation condition is vacuous. Otherwise, t^+ is valent at t by Proposition 2.28, so $|t^+| = \lambda_t$ by Lemma 2.11. Hence $F_{\mu,t}$ is finite in all cases.

Let $t' > t$, and suppose that $t \in A_{\nu,t'}$ for some $\nu < \mathrm{cf}_T(t')$. Note that t' must be a limit point. If it were isolated, then $t \notin A_{0,t'} = \{t'\}$.

Choose $s \in t^+$ such that $t' \in B_t(s)$. Since $t \in A_{\nu,t'} = (\phi_{t'}(\nu), t']$ and $\phi_{t'}(\nu) < t < s$, $s \in A_{\nu,t'}$. But $s \notin A_{\mu,t} \subset t^{\leq}$. Therefore

$$s \in F_{\mu,t} \cap (A_{\mu,t} \Delta A_{\nu,t'}).$$

Theorem 5.9 then yields a binary generator for T. □

Corollary 5.22. *Every non-zero ordinal endowed with the upper-limit topology has a binary generator.*

Proof. Let $\alpha > 0$. With the upper-limit topology, α is a Hausdorff tree. Hence α is an immediately branching, MUB-valent, Hausdorff pseudotree. Moreover, for every $\beta < \alpha$, $\lambda_\beta \leq 1$. The result follows from Corollary 5.21. $\square$

We end with a negative example for the classification theorem.

Example 5.23. There exists a rooted Hausdorff tree that does not admit a binary generator.

Proof. Let $T = \omega^{\leq\omega}$ be the complete ω-ary tree of height $\omega + 1$ ordered by extension. For $s, t \in T$, $s \leq t$ if $\mathrm{dom}(s) \subset \mathrm{dom}(t)$ and $s(n) = t(n)$ for all $n \in \mathrm{dom}(s)$. The limit points of T are functions $t \in \omega^\omega$, and every limit point is maximal. Every point $t \in \omega^{<\omega}$ is isolated, and the empty function is the root of T. Moreover, since T is a tree with a root, T is MUB-valent and immediately branching by Corollary 2.44.

The tree T is also Dedekind complete. Let C be any bounded chain. If C is finite or contains a limit point $t \in \omega^\omega$, then the maximum of C is the least upper bound. If $C \subset \omega^{<\omega}$ is infinite, then the union of all the functions in C is a function in ω^ω and is the least upper bound. Hence, T is Hausdorff by Theorem 3.7.

Suppose, toward a contradiction, that there is a homogeneous pair $(\mathcal{M}, \mathcal{A})$ that finitely separates every point $t \in T$. For isolated points, let $F_{0,t}$ be the set guaranteed by finite separation.

Define $\psi : \omega^\omega \to \omega^{<\omega}$ by

$$\psi(t) := \min(A_{0,t}).$$

This map is well-defined. For each $t \in \omega^\omega$, $A_{0,t} \subset t^{\leq}$. Since $t^{\leq}$ is well-ordered, $A_{0,t}$ has a least element.

Moreover, $\psi(t) \in \omega^{<\omega}$. Since $\psi(t) \in A_{0,t}$, $\psi(t)$ and t are comparable. Since t is maximal, we must have $\psi(t) \leq t$. If $\psi(t) = t$, then $A_{0,t} = \{t\}$. However, this is not an open neighborhood of t, so $\psi(t) < t$.

Let $t \in \omega^{<\omega}$ with $\mathrm{dom}(t) = n$. Define

$$E_t := \omega \setminus \{s(n) : s \in t^+, \exists f \in F_{0,t}, s \leq f\}.$$

$|E_t| = \omega$ because each $f \in F_{0,t}$ determines at most one $s \in t^+$, and hence at most one $s(n) \in \omega$.

We build an element $x \in \omega^\omega$ recursively by defining a sequence of points $p_n \in \omega^{n+1}$ for every $n < \omega$.

For $n = 0$, choose $k \in E_\emptyset$. Define $p_0 \in \omega^1$ by $p_0(0) = k$. Suppose $p_n \in \omega^{n+1}$ has been defined. Choose $k \in E_{p_n}$. Then define $p_{n+1} \in \omega^{n+2}$ by $p_{n+1}(n + 1) = k$ and $p_{n+1}(i) = p_n(i)$ for every $i < n + 1$.

Define the limit point $x \in \omega^\omega$ by $x(n) = p_n(n)$. Consider the point $\psi(x) \in \omega^{<\omega}$. Let $n = \mathrm{dom}(\psi(x))$. Since $\psi(x) < x$, we have $\psi(x)(i) = x(i)$

for every $i < n$. Since $\psi(x) \in A_{0,x}$, finite separation guarantees a point

$$f \in F_{0,\psi(x)} \cap (A_{0,\psi(x)} \Delta A_{0,x}) = F_{0,\psi(x)} \cap (A_{0,x} \setminus \{\psi(x)\}).$$

Since $\psi(x)$ is the minimum of $A_{0,x}$ and $A_{0,x}$ is well-ordered, we have $\psi(x) < f$. Thus there exists $s \in \psi(x)^+$ such that $\psi(x) < s \leq f \leq x$. Hence $x(n) = s(n)$.

This contradicts the construction of x. The recursion chose $x(n) \in E_{\psi(x)}$, while the existence of $s \leq f$ gives $s(n) \notin E_{\psi(x)}$.

Thus no homogeneous pair $(\mathcal{M}, \mathcal{A})$ can finitely separate every point, and by Theorem 5.9, T has no binary generator. □

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